A Trichotomy for Height Counting Functions and Wide Spacing of Orbits in Arithmetic Dynamics

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Workshop on Recurrence, Transcendence, and Diophantine Approximation

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Height Counting and Orbit Spacing: A Tale of Two Topics

In this talk I will discuss two loosely related topics.

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$K/\mathbb{Q}$	a number field
X/K	a smoooth projective variety of dimension d such that $X(K)$ is Zariski dense in X.

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Topic #2: Let

$$f : X \longrightarrow X$$

be an endomorphism of X. How are the various orbits distributed within $X(K)$?

Height Functions Measure Arithmetic Complexity

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We fix a (very) ample divisor D on X and an associated (absolute logarithmic) Weil height function

$$h_X : X(\bar{K}) \longrightarrow \mathbb{R}_{\geq 0}.$$

The details do not matter, it is enough for our purposes to know that $h_X(P)$ measures the *arithmetic complexity* of $P \in X(K)$, in the sense that

$$h_X(P) \approx \begin{pmatrix} \text{the number of bits required} \\ \text{to store } P \text{ on a computer.} \end{pmatrix}$$

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Observation: For a given bound T, there are the only finitely many points $P \in X(K)$ satisfying $h_X(P) \leq T$.

Height Counting Functions

We measure the density of points in $X(K)$ using the **height counting function**

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Examples: For projective space, abelian varieties, and curves of high genus:

$$\begin{aligned} \mathsf{N}\bigl(\mathbb{P}^N(K),T\bigr) &\approx T^{N+1}. \\ \mathsf{N}\bigl(A(K),T\bigr) &\approx (\log T)^{\operatorname{rank} A(K)}. \\ \mathsf{N}\bigl(C(K),T\bigr) &= O(1) \text{ for } g(C) \ge 2. \end{aligned}$$

Height Density Trichotomy Conjecture

Informal Conjecture: There are only three possible growth rates!

$$\mathsf{N}\big(X(K), T\big) \quad \begin{cases} \text{grows like a power of } T\text{, or} \\ \text{grows like a power of } \log T\text{, or} \\ \text{is bounded at } T \to \infty. \end{cases}$$

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There are problems with this formulation.

- We want to ignore subvarieties with "too many" points. For example, if $\tilde{X}$ is X blown up at a point, then

$$\mathsf{N}\big(\tilde{X}(K),T\big) \ge \mathsf{N}\big(\mathbb{P}^{d-1}(K),T\big).$$

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- We may need to take a finite extension of K to get generic behavior, e.g., $x^2+y^2+z^2=0$ over $\mathbb{Q}$.
- "Power of T" should mean something like

$$T^a \ll \mathsf{N}\big(X(K),T\big) \ll T^b; \quad \text{ditto "power of } \log(T)\text{."}$$

Iterated Logarithms and Growth Rates

We denote the **iterated logarithm** by

$$\log^{(m)}(T) := \underbrace{\log \circ \log \circ \cdots \circ \log}_{m \text{ iterates}}(T).$$

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Example (Logarithmic Growth):

$$\log(T)^a \ll \mathsf{N}(T) \ll \log(T)^b$$

$$\Longrightarrow \quad \lim_{T\to\infty} \frac{\log^{(2)} \mathsf{N}(T)}{\log^{(3)} T} = 1.$$

The Arithmetic Order of a Variety: Attempt #1

Definition: The **arithmetic order of X** is the integer

$$\mathsf{m}(X) \ge 0$$

such that

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Intuition:

$$\begin{aligned}
\mathsf{m}(X) = 0 \quad &\Longrightarrow \quad \mathsf{N}\bigl(X(K), T\bigr) \asymp \text{Power of } T.\\
\mathsf{m}(X) = 1 \quad &\Longrightarrow \quad \mathsf{N}\bigl(X(K), T\bigr) \asymp \text{Power of } \log(T).\\
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where the supremum is over all finite extensions L/K.

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Problem: Maybe intermediate log growth, such as

$$\log^{(2)} \mathsf{N}\bigl(U(L),T\bigr) = (\log^{(2)} T)^{1+\epsilon}.$$

(Can this happen?) Solution is to take some more logs.

The Arithmetic Order of a Variety: Final Version

Definition: The **arithmetic order of X** is the integer

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such that for some $k \ge 2$,

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Notes:

- If there is a U such that $U(L)$ is finite for all L/K, we set
$$\mathsf{m}(X) = \infty.$$
- The definition of $\mathsf{m}(X)$ can be generalized to S-integral points on quasi-projective varieties [Ulm-1987], but for this talk I will stick with K-rational points on projective varieties.

The Height Density Trichotomy Conjecture

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(JS [Ulm-1987])
The arithmetic order exists and satisfies

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Conjectural Corollary. Let $U \subseteq X$. Then

$$\mathsf{N}\big(U(K), T\big) \ll \log\log T \quad \Longrightarrow \quad U(K) \text{ is finite.}$$

Informally: Gap Principle $\Longrightarrow$ Finiteness

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An Application: Mumford (1965) proved:

$$g(C) \geq 2 \quad \Longrightarrow \quad \mathsf{N}\big(C(K), T\big) \ll \log\log T.$$

Then the corollary says that $\#C(K)$ is finite, i.e., it implies the Mordell conjecture (Faltings' theorem).

The Trichotomy Conjecture

Informally, the conjecture says that on some Zariski open subset, there are the only three possible growth rates!

$$\mathsf{N}\big(X(K),T\big) \approx T^a \text{ or } (\log T)^b \text{ or bounded.}$$

The Trichotomy Conjecture and Distribution of Orbits

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Let

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It frequently happens that

$$h\big(f^n(P)\big) \asymp \delta^n \quad \text{for some } \delta > 1, \qquad (**)$$

and thus the orbit counting function grows very slowly,

$$\mathsf{N}\big(\mathcal{O}_f(P),\, T\big) \leq c \log\log(T). \qquad (1)$$

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Hence if the Trichotomy Conjecture $(*)$ is true, and if $(**)$ is true, and if $X(K)$ is Zariski dense, then

$$X(K) \smallsetminus \Big(\mathcal{O}_f(P_1) \cup \cdots \cup \mathcal{O}_f(P_r)\Big) \text{ is Zariski dense.}$$

Wide Spacing of Orbits

joint work with

Hector Pasten

Orbits of Rational Points and How They're Spaced

Notation:

$K/\mathbb{Q}$	a number field
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f	a morphism $f : X \to X$ defined over K
$\mathcal{O}_f(P)$	the (forward) orbit $\{f^n(P) : n \in \mathbb{N}\}$

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We start with a vague, but intriguing, principle that will be the primary theme for the rest of this lecture.

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Orbit Wide-Spacing Principle.
If $X(K)$ is Zariski dense in X, then the points in $X(K)$ lie in "*lots and lots*" of different "*widely spaced*" orbits.

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Caveat: We may need to replace K with a finite extension. But once we do that, then the Orbit Wide-Spacing Principle says that there is a number field over which X has lots of dynamically unrelated points.

What Makes a Set of Orbits "Widely Spaced"?

I will give two reasonable answers to this question (and several other answers are described in our paper). However, we first need to expand our orbits.

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Definition The **grand f-orbit of P** is the set of points whose f-orbits eventually merge with the orbit of P. Thus

$$\mathcal{O}_f^{\text{grand}}(P) = \{Q : f^n(Q) = f^m(P) \text{ for some } m, n \geq 0\}.$$

We will say that a grand orbit is **K-rational** if it contains a point of $X(K)$.

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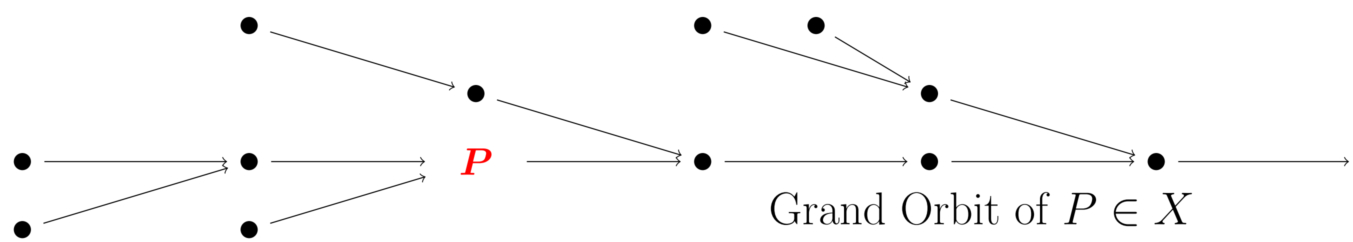

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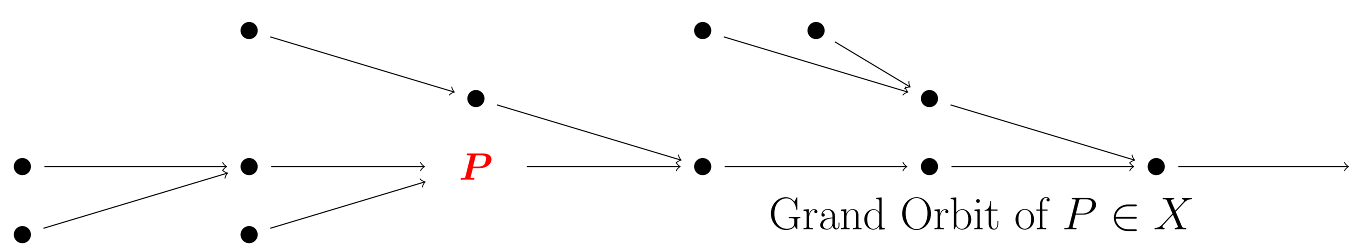

Observation: Grand f-orbits are disjoint, so $X(K)$ is the disjoint union of its grand f-orbits

Grand Orbits and Their Transversals

Definition: An ***f*-transversal for $X(K)$** is a subset $S \subseteq X(K)$ containing exactly one K-rational point in each K-rational grand orbit.

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Alternatively, we can define an equivalence relation $\sim_f$ on $X(K)$ corresponding to the grand f-orbit decomposition, and then a transversal S is a subset that contains exactly one point in each equivalence class, i.e.,

$$S \xleftrightarrow{\text{bijection}} X(K)/\sim_f \ .$$

The Orbit Transversal Game

We always assume that

$X(K)$ is Zariski dense.

The Orbit Transversal Game. In the game, first I replace K with a finite extension, then an f-transversal $S \subseteq X(K)$ is chosen. I win the game if S is Zariski dense.

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Now I need to tell you how the set S is chosen.

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There is always a winning f-transversal, i.e., there always exists a Zariski dense f-transversal for $X(K)$.

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Every f-transversal for $X(K)$ is a winning transversal, i.e., every f-transversal for $X(K)$ is Zariski dense.

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- Weak Transversality says that I can win the game by carefully choosing one point from each K-rational grand orbit.

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- Weak Transversality says that I can win the game by carefully choosing one point from each K-rational grand orbit.
- Strong Transversality says that I win even if I allow you to choose the points in the K-rational grand orbits.

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- Weak Transversality says that I can win the game by carefully choosing one point from each K-rational grand orbit.
- Strong Transversality says that I win even if I allow you to choose the points in the K-rational grand orbits.
- Both transversality properties are a way of quantifying the idea that the grand orbits are "widely spaced."

An Alternative Formulation of Weak Transversality

Reduction Lemma. Assume that $X(K)$ is Zariski dense in X. Then the following are equivalent:

- There exists a Zariski dense f-transversal for $X(K)$, i.e., weak transversality is true.
- For every finite collection of grand orbits $\Gamma_1, \ldots, \Gamma_r$ in X, the set

$$X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r) \quad \text{is Zariski dense in } X.$$

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$$X(K) \smallsetminus (\Gamma_1 \cup \cdots \cup \Gamma_r) \quad \text{is Zariski dense in } X.$$

Remark about the proof: The proof of the reduction lemma relies on an enumeration of the Zariski closed subsets of X/K. It thus uses the fact that K is countable.

Orbit Wide-Spacing Theorems

Theorem. (HP–JS)

(a) Weak Orbit Transversality is true for:

  (1) $\mathbb{P}^N$ and $\deg(f) \geq 2$.

  (2) K3 surfaces.

(b) Strong Orbit Transversality is true for:

  (1) $\mathbb{P}^N$ and $\deg(f) = 1$.

  (2) Geometrically simple abelian varieties.

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(2) Geometrically simple abelian varieties.

Tools used in the proofs include:

- Canonical (and non-canonical) heights
- Linear algebra/linear recurrence calculations
- Classical algebraic geometry
- p-adic methods
- Classification of rational points on subvarieties of abelian varieties (Faltings et al.)
- Classification of orbit points on subvarieties for étale maps (Bell, Ghioca, Tucker)

Proof Sketches (As Time Permits)

- Weak transversality for $\mathbb{P}^N$ and $\deg(f) \geq 2$
- Strong transversality for $\mathbb{P}^N$ and $\deg(f) \geq 1$
- Weak transversality for K3 surfaces

Weak Transversality for $X = \mathbb{P}^N$ and $\deg(f) = d \geq 2$

Proof Ideas/Sketch:

(1) The height function

$$h : \mathbb{P}^N(K) \longrightarrow \mathbb{R}_{\geq 0},$$
$$h(P) = \bigl(\text{\# of bits required to store } P \text{ on a computer}\bigr)$$

satisfies

$$h\bigl(f(P)\bigr) = d \cdot h(P) + O(1).$$

(2) We can get rid of the $O(1)$ by defining the ***f*-canonical height** of P as the limit

$$\hat{h}_f(P) = \lim_{n\to\infty} \frac{1}{d^n} h\bigl(f^n(P)\bigr).$$

The canonical height satisfies

$$\hat{h}_f = h + O(1), \qquad \hat{h}_f\bigl(f(P)\bigr) = d\hat{h}_f(P),$$
$$\hat{h}_f(P) = 0 \iff \mathcal{O}_f(P) \text{ is finite.}$$

Weak Transversality for $X = \mathbb{P}^N$ and $\deg(f) = d \geq 2$

(3) Let $\Gamma \subseteq \mathbb{P}^N(K)$ be an infinite grand orbit. Then

$$\hat{h}^{\min}(\Gamma) := \inf_{P \in \Gamma} \hat{h}_f(P) > 0.$$

The positivity follows from $\hat{h}_f = h + O(1)$ and the fact that there are only finitely many points of bounded height.

(4) Taking an arbitrary $Q \in \Gamma$ and relating it to a point of minimal height in Γ, we deduce that there is an $n(Q) \geq 0$ such that

$$\hat{h}_f(Q) = d^{n(Q)} \cdot \hat{h}_f^{\min}(\Gamma).$$

Then $\hat{h}_f = h + O(1)$ tells us that

$$h(Q) = d^{n(Q)} \cdot \hat{h}_f^{\min}(\Gamma) + \underbrace{O(1)}_{\text{depends only on } f}. \qquad (*)$$

Weak Transversality for $X = \mathbb{P}^N$ and $\deg(f) = d \geq 2$

(5) Hence for grand orbits $\Gamma_1, \ldots, \Gamma_r$,

$$\bigcup_{i=1}^{r} \Big\{ h(Q) : Q \in \Gamma_i \Big\} \subseteq \bigcup_{i=1}^{r} \bigcup_{n \in \mathbb{N}} \Big[a(i) \cdot d^n - b,\ a(i) \cdot d^n + b \Big]$$

An elementary estimate shows that the double union omits intervals in $\mathbb{R}_{\geq 0}$ of arbitrarily large length, and hence (with some work) it omits enough values in

$$\Big\{ h(Q) : Q \in \mathbb{P}^N(\mathbb{Q}) \Big\}$$

to prove that $\Gamma_1 \cup \cdots \cup \Gamma_r$ misses a Zariski dense subset of $\mathbb{P}^N(\mathbb{Q})$. □

N.B. It is vital that $(*)$ uses the canonical height $\hat{h}_f$.

Strong Transversality for $X = \mathbb{P}^N$ and $\deg(f) = 1$

Proof Ideas/Sketch:

(1) Replacing K by a finite extension, we change coordinates so that f is given by a matrix in Jordan normal form. An easy argument deals with the case that an eigenvalue of f is 0 or a root of unity.

(2) Let $S \subset \mathbb{P}^N(K)$ be an f-transversal set, so

$$\left\{\mathcal{O}_f^{\text{grand}}(P) : P \in S\right\} = \left\{\begin{matrix}\text{distinct grand} \\ \text{orbits in } \mathbb{P}^N(K)\end{matrix}\right\}.$$

Goal: Show S is Zariski dense.

(3) Let $\phi(\boldsymbol{x}) \in K[\boldsymbol{x}]$ satisfy

$$\phi(Q) = 0 \quad \text{for all } Q \in S.$$

Goal: Show $\phi = 0$.

Strong Transversality for $X = \mathbb{P}^N$ and $\deg(f) = 1$

(4) Let $\mathfrak{p}$ be a (good) prime, with uniformizer π, let

$$r_i = (1 + \deg \phi)^{N-i} \quad \text{for } 0 \le i < N,$$

and consider at the points

$$P_{\boldsymbol{r}} := [\pi^{-r_0}, \pi^{-r_1}, \ldots, \pi^{-r_N}].$$

(5) Since S is an f-transversal set, we can find

$$Q_{\boldsymbol{r}} \in S \text{ with } \mathcal{O}_f^{\text{grand}}(Q_{\boldsymbol{r}}) = \mathcal{O}_f^{\text{grand}}(P_{\boldsymbol{r}}).$$

Then

$$Q_{\boldsymbol{r}} = f^{n_{\boldsymbol{r}}}(P_{\boldsymbol{r}}) \quad \text{for some } n_{\boldsymbol{r}} \in \mathbb{Z}.$$

A computation shows that the valuations of the non-zero monomials in

$$\phi(Q_{\boldsymbol{r}}) = \phi\big(f^{n_{\boldsymbol{r}}}(P_{\boldsymbol{r}})\big) = 0$$

are distinct. Hence $\phi = 0$. $\square$

Weak Orbit Transversality for K3 Surfaces

Proof Ideas/Sketch: Let X/K be a smooth projective K3 surface defined over a number field such that $X(K)$ is Zariski dense, and let

$$f : X \longrightarrow X$$

be an endomorphism of X. In the following, we may replace K by a finite extension.

(1) The endomorphism f is an automorphism.

Proof. The fact that $\mathcal{K}_X = \mathcal{O}_X$ and $f^*\mathcal{K}_X = \mathcal{K}_x \otimes \mathcal{R}_f$ imply that f is étale, and then the fact that $\pi_1(X(\mathbb{C})) = 1$ implies that f is an automorphism.

(2) If f has finite order, then grand orbits are finite So we may assume that f has infinite order.

(3) Grand orbits for automorphisms break up into

$$\mathcal{O}_f^{\text{grand}}(P) = \mathcal{O}_f(P) \cup \mathcal{O}_{f^{-1}}(P),$$

so it suffices to prove that the complement of finitely many forward orbits is Zariski dense.

Weak Dense Orbit Transversal Conjecture for K3 Surfaces

(4) There exists a *rational* curve $C \subset X$ that is not f-periodic.

We use two facts in the literature:
(a) Non-trivial effective divisors on K3 surfaces are linearly equivalent to sums of rational curves. (Li–Liedtke 2012)
(b) If $g \in \mathrm{Aut}(X)$ and $H \in \mathrm{Div}(X)$ is ample and $g_*H \sim H$, then g has finite order.
Proof of (4). Let D be ample effective, write as sum $D \sim \sum n_i C_i$ with C_i rational. If all C_i are f-periodic, then $f_*^k D \sim D$ for some $k \geq 1$, contradicting f has infinite order.

(5) Extending K, we may assume that $\overline{C(K)} = C$.

(6) For every $n \in \mathbb{Z}$ and every $Q \in X$, the set

$$\mathcal{O}_f(Q) \cap f^n(C) \quad \text{is finite.}$$

Hence

$$f^n\big(C(K)\big) \smallsetminus \mathcal{O}_f^{\text{grand}}\big(\{Q_1,\dots,Q_r\}\big) \quad \text{is Zariski dense in } C(K).$$

Proof. The Dynamical Mordell–Lang Conjecture implies that for an endomorphism $f : V \to V$ and a non-periodic curve $W \subset V$, the intersection $\#\big(\mathcal{O}_f(Q) \cap W\big)$ is finite. The DML is true for étale maps (Bell–Ghioca–Tucker 2010), so in particular it is true for automorphisms.

(7) Combining (6) with the fact that the union of the curves $f^n(C)$ is Zariski dense in X completes the proof. □

Selected References

[Ulm-1987] J.H. Silverman, Integral points on curves and surfaces, Proc. 15$^{\text{th}}$ Journées Arithmétiques, Ulm, 1987, *Lect. Notes in Math.* **1380** (1989), 202–241.

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In Conclusion

I would like to thank you for your attention, and to thank the organizers for inviting me to attend and speak at this workshop.

But most importantly . . . .

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A Trichotomy for Height Counting Functions and Wide Spacing of Orbits in Arithmetic Dynamics

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