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Large zeros of Linearly Recurrent Sequences

Florian Luca
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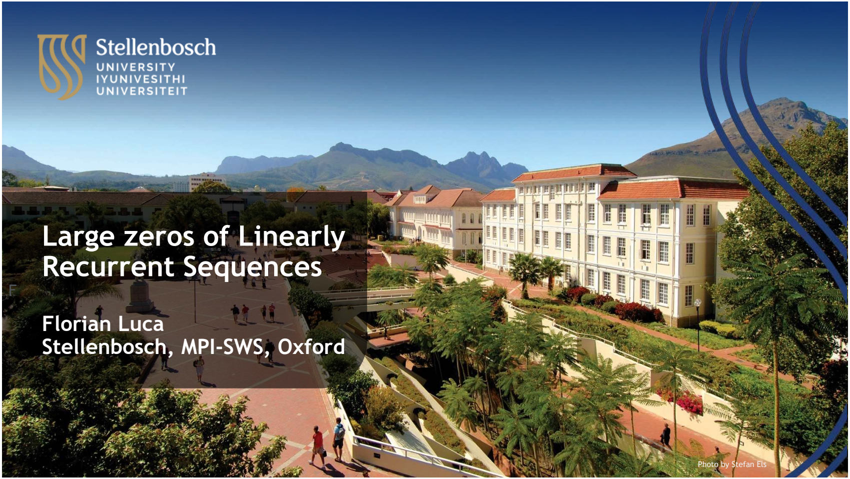

Photo by Stefan Els

Linearly recurrent sequences

A **linear recurrent sequence (LRS)** is a sequence in $\mathbb{Z}$ (or $\mathbb{Q}$) $\langle u_0, u_1, u_2, \ldots \rangle$ such that there are constants $a_1, \ldots, a_k$ and,
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One can write

$$u_n = \sum_{j=1}^{k} P_j(n) \lambda_j^n \qquad \forall n \geq 0,$$

with $P_j(x) \in \mathbb{C}[x]$, where the above data can be read from the recurrence and initial values.

The Skolem Problem

Problem SKOLEM

Instance: A linear recurrence sequence $\langle u_0, u_1, u_2, \ldots \rangle$

Question: Does $\exists n \geq 0$ such that $u_n = 0$?

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This problem has been open for about 90 years.

The Skolem-Mahler-Lech Theorem

Theorem (Skolem 1934; Mahler 1935, 1956; Lech 1953)

The set of zeros of a linear recurrence sequence is semi-linear:

$$\{n : u_n = 0\} = F \cup A_1 \cup \ldots \cup A_\ell$$

where F is finite and each A_i is a full arithmetic progression.

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Neither is effective

Bounds on the number of zeros

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Such sequences can be written as merge of lower-order LRS's (e.g., $\langle 1, 2, 1, 2, 1, 2, \ldots \rangle$).

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MSTV property: at most 3 dominant roots w.r.t. usual absolute value on $\mathbb{C}$ and at most 2 dominant roots w.r.t. p-adic absolute value.

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$$u_{n+5} = 9u_{n+4} - 10u_{n+3} + 522u_{n+2} - 4745u_{n+1} + 4225u_n$$

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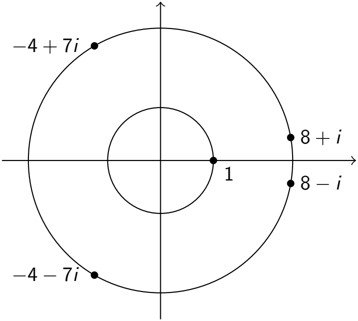

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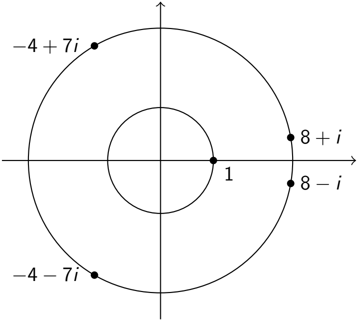

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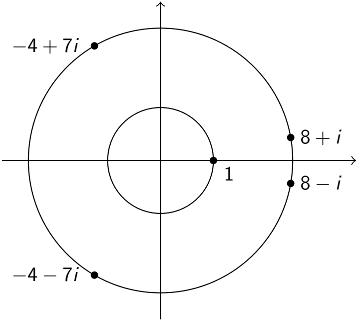

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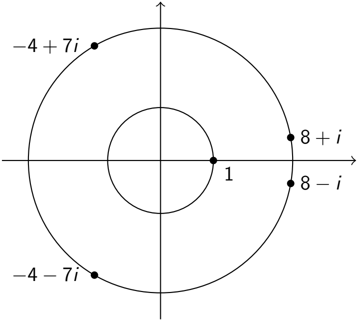

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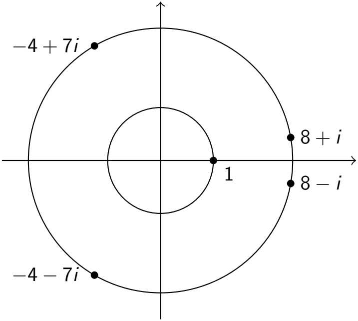

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- **Empirically**: for all $u_0, \ldots, u_4$ there exists k s.t. $u_n = 0 \pmod{5^k}$ for only finitely many n. **Why??**

A **universal Skolem set (USS)** is a subset $\mathcal{S}$ of $\mathbb{N}$ such that for all linearly recurrent sequences $\langle u_n \rangle$ the set

$$\{n \in \mathcal{S} : u_n = 0\}$$

is computable.

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"Can one solve the Skolem Problem on the set of primes? "

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Theorem (L., Ouaknine, Worrell, 2021)

Define $f : \mathbb{N} \setminus \{0\} \to \mathbb{N}$ *by*

$$f(n) := \lfloor \sqrt{\log n} \rfloor,$$

and define the sequence $(s_n)_{n \geq 0}$, *inductively by*

$$s_0 = 1 \qquad \text{and} \qquad s_n = n! + s_{f(n)} \quad \text{for} \quad n > 0.$$

Then $\mathcal{S} := \{s_n : n \in \mathbb{N}\}$ *is a universal Skolem set.*

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The first few elements of $\mathcal{S}$ are

$$\{1, 1! + 1, 2! + 1, 3! + 1, 4! + 1, 5! + 1, 6! + 1, 7! + 1, 8! + 2! + 1, \ldots\}$$

or

$$\{1, 2, 37, 25, 121, 721, 5041, 40323, \ldots\}.$$

LICS 2021 Distinguished Papers

Fusible numbers and Peano Arithmetic – Jeff Erickson, Gabriel Nivasch and Junyan Xu.

Positive first-order logic on words – Denis Kuperberg.

Inapproximability of Unique Games in Fixed-Point Logic with Counting – Jamie Tucker-Foltz. (co-winner of Kleene Award for Best Student Paper)

Separating Rank Logic from Polynomial Time – Moritz Lichter. (co-winner of Kleene Award for Best Student Paper)

Lacon- and Shrub-Decompositions: A New Characterization of First-Order Transductions of Bounded Expansion Classes – Jan Dreier.

A Logic for Locally Complete Abstract Interpretations – Roberto Bruni, Roberto Giacobazzi, Roberta Gori and Francesco Ranzato.

Orbit-Finite-Dimensional Vector Spaces and Weighted Register Automata – Mikołaj Bojańczyk, Bartek Klin and Joshua Moerman.

Universal Skolem Sets – Florian Luca, Joel Ouaknine and James Worrell.

Compositional Semantics for Probabilistic Programs with Exact Conditioning – Dario Stein and Sam Staton.

Minimal Taylor Algebras as a Common Framework for the Three Algebraic Approaches to the CSP – Libor Barto, Zarathustra Brady, Andrei Bulatov, Marcin Kozik and Dmitriy Zhuk.

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$$\#(\mathcal{S} \cap [1, x]) = (1 + o(1)) \frac{\log x}{\log \log x} \quad \text{as} \quad x \to \infty.$$

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Highly representable integers

Definition

A **representation** of $n \in \mathbb{N}$ is a triple (P, q, a) such that

$$n = Pq + a,$$

where

- P is prime;
- $q \in Q := [\log\log n, \sqrt{\log n}]$;
- $a \in A := \left[\dfrac{\log n}{2\sqrt{\log\log\log n}}, \dfrac{\log n}{\sqrt{\log\log\log n}}\right]$.

Let $r(n)$ denote the number of representations of n. We say that $n > 10^{10}$ is **highly representable** if

$$r(n) > \log\log\log\log n.$$

Highly representable integers: Examples

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Example

For example, $n = 10^{1000} + k$ is **highly representable** for $k \in [0, 1000]$ exactly for

$$k \in \{161, \ldots, 248\} \cup \{325, \ldots, 553\} \cup \{606, \ldots, 730\}.$$

Hint: For all the above n we have

$$Q = [6, 15], \qquad A = [89, 176], \qquad \log\log\log\log n \approx 0.52.$$

A thicker universal Skolem set

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Theorem (L., Maynard, Noubissie, Ouaknine, Worrell 2023)

Let $\mathcal{S}$ be the set of highly representable n's. Then

- *$\mathcal{S}$ is a universal Skolem set*
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So, this does not yet solve the Skolem problem.

Large zeros of LRS's

Definition

For a nondegenerate LRS $\mathbf{u} = \langle u_n \rangle_{n \geq 0} \subset \mathbb{Z}$ given by

$$u_{n+k} = a_1 u_{n+k-1} + \cdots + a_k u_n \qquad \forall n \geq 0,$$

let its size be

$$C_{\mathbf{u}} := \max\{k, |a_1|, \ldots, |a_k|, |u_0|, \ldots, |u_{k-1}|, 12\}.$$

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Given a nondegenerate LRS $\mathbf{u} \subset \mathbb{Z}$, we say that n is a **large zero** of $\mathbf{u}$ if

$$u_n = 0 \qquad \text{and} \qquad n > 2\exp_6(C_{\mathbf{u}}).$$

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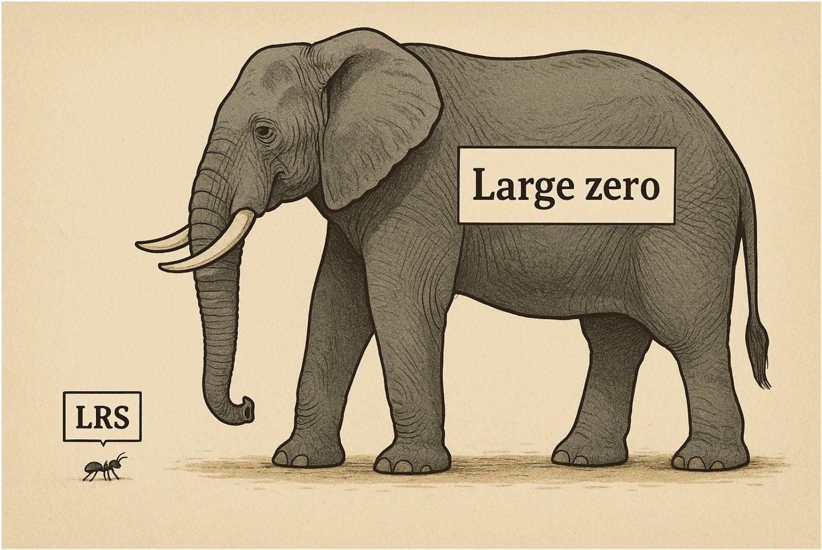

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Theorem (L., Ouaknine, Worrell, 2025)

The set $\mathcal{L}$ has zero density. In fact, writing $\mathcal{L}(X) = \mathcal{L} \cap [0, X]$, the inequality

$$\#\mathcal{L}(X) = O\left(\frac{X}{(\log X)^B}\right)$$

holds with any $B > 0$ for all $X \geq 2$.

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- There are very few such $\mathbf{u}$.
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- Now conclude.

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- That is, we conjecture that there is no large zero of any nondegenerate LRS.
- In the rest of the talk, I would like to bring some heuristic arguments to support the above conjecture.
- More surprisingly, a classical conjecture concerning the distribution of primes seems to have something to do with the above conjecture.

Some notations

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For a positive integer m let

$$v_{\sigma,m} := \sum_{j=1}^{s} P_j(m)\gamma_j\lambda_j^m. \tag{2}$$

Good and bad primes

Good and bad primes

Definition

We say that $P \in [X, 2X]$ is **bad** if there exists:

(i) a nondegenerate LRS $\mathbf{u}$ given by which is small at level X; i.e.,

$$C_{\mathbf{u}} < \log_6 X,$$

(ii) a permutation σ of $\{1, \ldots, s\}$,

(iii) a positive integer $m \in [1, X^{1/4}]$,

such that

(1) The number $v_{\sigma,m}$ shown at (2) is nonzero and

(2) P divides the numerator of

$$N_{\mathbb{K}/\mathbb{Q}}(v_{\sigma,m}).$$

Counting bad primes

Let $\mathcal{P}_{\mathrm{bad}}(X)$ be the set of bad primes in $[X, 2X]$.

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Theorem (L., Ouaknine, Worrell 2025)

We have

$$\#\mathcal{P}_{\mathrm{bad}}(X) < X^{2/3}$$

for all $X > X_0$.

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Now we sum up over all the $X^{1/4+o(1)}$ possibilities for $(\mathbf{u}, \sigma, m)$ getting a bound of $X^{1/2+o(1)}$ on $\#\mathcal{P}_{\mathrm{bad}}(X)$.

Primes in short intervals

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- Ford, Green, Konyagin, Tao and Maynard showed that

$$p_{n+1}-p_n \gg \frac{\log p_n \log\log p_n \log\log\log\log p_n}{\log\log\log p_n} \qquad \text{i. o.}$$

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- So, taking $k = \lfloor \kappa(\log n)^2 \rfloor$ with some $\kappa > 1$, this is

$$\left(1 - \frac{1}{\log n}\right)^{\kappa(\log n)^2} = \frac{1}{n^{\kappa}} = o\left(\frac{1}{n}\right)$$

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- So, taking $n \in [X, 2X]$, the probability of all the above happening is
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but the above interval has length X, so maybe there is no such n in $[X, 2X]$ for large X.

- The model is wrong since the events "n is composite" and "$n+1$ is composite" are not independent (one of them is always even).

A modified Cramér conjecture

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Since $\#\mathcal{P}_{\rm bad}(X) < X^{2/3}$ for $X > X_0$ it follows that asymptotically $\mathcal{P}\backslash\mathcal{P}_{\rm bad}$ has the same counting function as the primes.

A modified Cramér conjecture

Since $\#\mathcal{P}_{\mathrm{bad}}(X) < X^{2/3}$ for $X > X_0$ it follows that asymptotically $\mathcal{P} \backslash \mathcal{P}_{\mathrm{bad}}$ has the same counting function as the primes.

Let q_n be the $n$the element of $\mathcal{P} \backslash \mathcal{P}_{\mathrm{bad}}$.

Conjecture (Modified Cramér conjecture)

Assume that there exists $\kappa > 0$ such that

$$\limsup_{n \to \infty} \frac{q_{n+1} - q_n}{(\log q_n)^2} = \kappa.$$

Solving the Skolem problem conditionally

Theorem (L., Ouaknine, Worrell 2025)

The modifed Cramér conjecture implies that there exists an absolute constant n_0 such that a nondegenerate LRS $\mathbf{u}$ has no large zeros $n > n_0$. That is, if $u_n = 0$, then

$$n < \max\{n_0, C_{\mathbf{u}}\}.$$

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- This implies

$$v_{\sigma,m} \equiv 0 \pmod{P},$$

where σ is the Frobenius with respect to P (so $\gamma_j \equiv \lambda_j^P \pmod{P}$).

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- Putting it together we get $n = O(1)$.


Thank you very much
Hartelijk dank
Stellenbosch
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Photo by Stefan Els