The Tate–Voloch Conjecture and applications to the arithmetic of abelian varieties

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For any fixed $a_1, \ldots, a_n \in \mathbb{C}_p$, there is a constant $c > 0$ such that for any roots of unity $\zeta_1, \ldots, \zeta_n \in \mathbb{C}_p$, either $\sum_i a_i \zeta_i = 0$ or $|\sum_i a_i \zeta_i|_p > c$.

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From this theorem is it easy to deduce:

Corollary

Let $f \in \mathbb{C}_p[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ be a Laurent polynomial defining a hypersurface in $\mathbb{G}_m^n$. There is a $c > 0$ such that for any roots of unity $\zeta_1, \ldots, \zeta_n$, either $f(\zeta_1, \ldots, \zeta_n) = 0$ or $|f(\zeta_1, \ldots, \zeta_n)|_p > c$.

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Consequently, given any closed subvariety X of $\mathbb{G}_m^n$, there is a $c_X > 0$ such that for any torsion point $P \in \mathbb{G}_m^n$, either $P \in X$ or $d(P, X) > c_X$.

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Let G be a semiabelian variety over $\mathbb{C}_p$, and let X be a closed subvariety of G (also defined over $\mathbb{C}_p$). There is an $\epsilon > 0$ such that any $P \in G(\mathbb{C}_p)_{\mathrm{tors}}$ satisfies $d(P, X) > \epsilon$ or $P \in X$.

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This is a refinement of the discreteness of torsion.

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E.g., Distance of roots of unity to 1 in $\mathbb{G}_m$ over $\mathbb{C}_p$:

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E.g., Distance of roots of unity to 1 in $\mathbb{G}_m$ over $\mathbb{C}_p$:

When $\zeta^n = 1$ for $n \geq 2$ and $(n, p) = 1$, we have $|\zeta - 1|_p = 1$, and when ζ is a primitive p^k-th root of unity,

$$|\zeta - 1|_p = p^{-1/(p^k - p^{k-1})}.$$

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Alternatively, can use ultrafields. Let ω be a non-principal ultrafilter, and let I_ω be the ideal of the ring R of sequences in $\overline{K}$ defined by

$$(a_i)_{i\in\mathbb{N}} \in I_\omega \quad \Longleftrightarrow \quad \forall n \in \mathbb{N},\ \{i : |a_i|_p \leq p^{-n}\} \in \omega.$$

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- Corpet (2013) proved T-V using Galois theory and ultrafields, for the case when both G and X are defined over $\overline{\mathbb{Q}_p}$.
- Scanlon's and Corpet's approaches have a lot of philosophical overlap with Hrushovski's proof of Manin–Mumford: they embed the Diophantine question into a setting with richer logic or symmetry (e.g., difference fields or ultraproducts), where rigidity forces the arithmetic conclusion.

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Suppose $\{P_n\}_{n=1}^{\infty} \subseteq G_{\mathrm{tors}}$ converges p-adically to X as $n \to \infty$.

Form the projection to the quotient $\pi : G \to G/B$, noting that $\pi(a)$ and the $\pi(P_n)$ are torsion.

$$(P_n) \to X \Longrightarrow (\pi(P_n)) \to \pi(a)$$

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$$(P_n) \to X \implies (\pi(P_n)) \to \pi(a)$$

But discreteness of torsion in G/B then implies that $\pi(P_n) = \pi(a)$ for all sufficiently large n, so

$$P_n - a \in B \quad \forall n \gg 0$$

and

$$P_n \in X \quad \forall n \gg 0.$$

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Proof sketch:

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1. Produce a difference equation $F(\sigma) = 0$ satisfied by all torsion points of G, where none of the roots of F are roots of unity.
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2. From X and the assumption $(P_n) \to X$, produce a special subvariety Z of G^d, where $d = \deg(F)$.
3. $\{P_n\}_{n=1}^{\infty}$ yields a sequence $\{P_{n,\sigma}\}_{n=1}^{\infty}$ tending to Z, to which we apply T-V for special subvarieties.

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In general, for $F = a_d T^d + \cdots + a_0 \in \mathbb{Z}[T]$ and $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, we write

$$F(\sigma)(P) = a_d \sigma^d(P) + \cdots + a_1 \sigma(P) + a_0,$$

where $a_i \sigma^i(P) = \underbrace{\sigma^i(P) + \cdots + \sigma^i(P)}_{a_i \text{ times}}$ for $+$ the group law.

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Let $\ell \neq p$ be prime, and let $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ be such that

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Thus $F(T)(\sigma) = (T-\ell)(T-p)(\sigma)$ vanishes on all roots of unity, and so $F(T)(\sigma)$ vanishes on

$$\{(\zeta_1,\ldots,\zeta_k) : \zeta_i^{n_i} = 1 \text{ for some } n_i \in \mathbb{Z}_+\} = G_{\mathrm{tors}}.$$

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From $F(\sigma(P_n)) = 0$, we obtain

$$\sigma(P_{n,\sigma}) = (\sigma(P_n), \sigma^2(P_n)) = M(P_{n,\sigma}).$$

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As X is defined over $\mathbb{Q}_p$, hence is Galois-stable, this gives

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and thus

$$P_{n,\sigma} \to M^{*,r}(X^2) \quad \forall r.$$

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Since the sequence $\{P_{n,\sigma}\}$ defines an ultrafield-valued point of G^2 lying in $M^{*,r}(X^2)$ for each r, the intersection

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Since the sequence $\{P_{n,\sigma}\}$ defines an ultrafield-valued point of $M^{-r}(X^2)$ for each r, the intersection

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And clearly

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$$Z := \bigcap_{m \geq 0} M^m(Y)$$

is nonempty, closed, and satisfies

$$M(Z) = Z.$$

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As none of the roots of F are roots of unity, so that M^s is not acting as an automorphism of finite order on Z', we conclude that Z' is a special subvariety of G^2.

Step 3: conclude T-V for X

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Projecting to the first coordinate of G^2, we get $P_{n_j} \in X$ for all $j \gg 0$. $\square$

Remarks

- Mixed characteristic is easier to handle than non-mixed (can use Frobenii and also avoid separability issues in Galois extensions)
- prime-to-p torsion is easier to handle than p-primary torsion
- In general one doesn't find difference equations killing the torsion points themselves, but rather, difference equations killing ultrafield-valued points P^* of X arising from sequences of torsion points.

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If T-V holds for subvarieties of A/K_v (in particular wrt D), then

$$\lim_{n\to\infty} \frac{1}{\#\mathrm{Gal}(\overline{K_v}/K_v)\cdot x_n} \sum_{y\in\mathrm{Gal}(\overline{K_v}/K_v)\cdot x_n} \lambda_{D,v}(y) = \int \lambda_{D,v}(y)\,d\mu_v = 0.$$

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- Ih's conjecture is a theorem for elliptic curves (Baker–Ih–Rumely '07).
- There are "parameter-space" versions of Ih's conjecture in dimension 1 (cf. work by Benedetto–Ih).

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OTOH, the diophantine approximation theorem of David–Hirata-Kohno proves logarithmic equidistribution at archimedean places. □

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- Work in progress (with R. de Jong and F. Shokrieh): adapt this argument to abelian varieties over global function fields. Proof rests on adaptation of Corpet's proof of T-V to completions of global function fields.