Fragments of Hilbert's Program

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(joint work with Valérie Berthé, Florian Luca, James Worrell,
Toghrul Karimov, Joris Nieuwveld, Mihir Vahanwala)

Workshop on recurrence, transcendence, and Diophantine approximation
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Funded by DFG Deutsche Forschungsgemeinschaft German Research Foundation

Calculemus!

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Calculemus!

If I have seen farther than others, it is because I have stood on the shoulders of giants;
You, my dear Hooke, have not.

Leibniz's Dream: reducing mathematics to mere computation

Leibniz visited the Royal Society in 1673 where he demonstrated a calculating machine that he had designed and had been building since 1670. The machine was able to execute all four basic operations (adding, subtracting, multiplying, and dividing), and the Royal Society promptly elected him as external member.

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``In large language models (LLMs), a *token* is a chunk of text used as a basic unit for processing -- typically a word, subword, or even character, depending on the language and tokenizer. The model reads and generates text in terms of these tokens, not full words or sentences, enabling it to handle diverse languages and structures efficiently.''

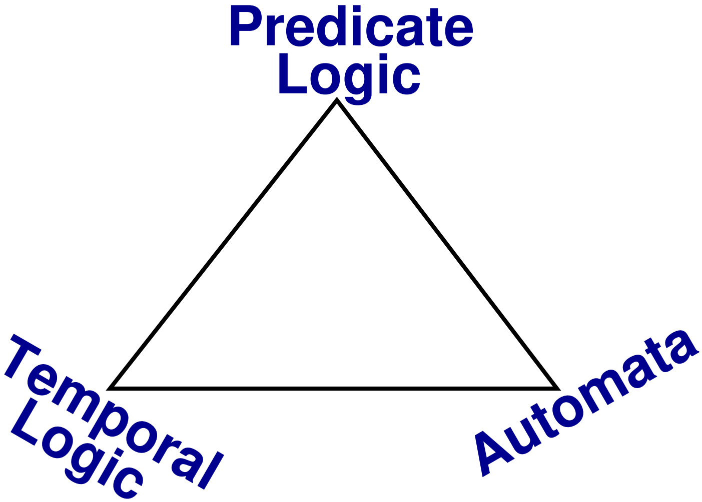
Hilbert's Program and the Entscheidungsproblem

Hilbert's Program and the Entscheidungsproblem

Wir müssen wissen. Wir werden wissen.

Fast forward about a hundred years. . .

The Holy Trinity of Theoretical Computer Science



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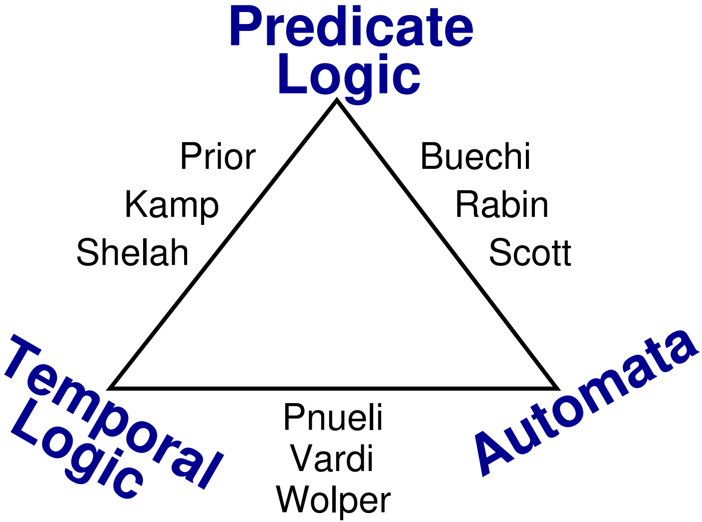

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Let P denote the set of prime numbers. Then the following holds:

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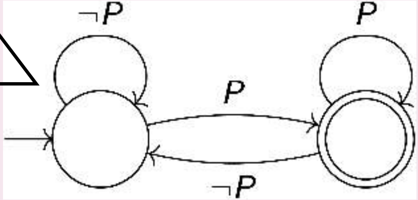
$$\forall x \, \exists y \, (y > x \wedge P(y))$$

$$\mathbf{G} \, \mathbf{F} \, P$$



The Monadic Second-Order Logic of Order

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The syntax of **MSO**$\langle \mathbb{N}; < \rangle$

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$$\varphi ::= x < y \mid P(x) \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi \mid \exists P\, \varphi$$

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Theorem (Dirichlet, 1837)

Let PRIMES denote the set of prime numbers. Then

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- Above formula is an example of a sentence in the MSO theory of $\langle \mathbb{N}; <, PRIMES \rangle$

Question (Büchi and Landweber, 1969)

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Theorem (Bateman, Jockusch, and Woods, 1993)

Yes, assuming Schinzel's Hypothesis H.

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Are there finitely many n such that the number of perfect squares between 2^n and 2^{n+1} is even, *and* the number of perfect squares between 2^{n+1} and 2^{n+2} is divisible by 3?

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Solution requires Baker's theorem on linear forms in logarithms!

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. . . much subsequent work over the ensuing decades (Landweber, Semenov, Thomas, Rabinovich, Carton, . . .)

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RESEARCH-ARTICLE | OPEN ACCESS

On the Decidability of Monadic Second-Order Logic with Arithmetic Predicates

Authors: Valérie Berthé, Toghrul Karimov, Joris Nieuwveld, Joël Ouaknine, Mihir Vahanwala, and James Worrell | Authors Info & Claims

LICS '24: Proceedings of the 39th Annual ACM/IEEE Symposium on Logic in Computer Science

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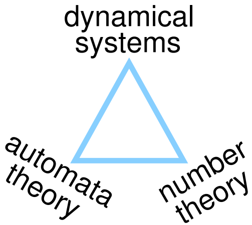

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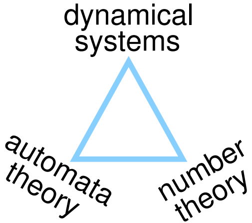

We are planning to implement this algorithm!

A glimpse of the Hilbert Landscape

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The first pair is
$(m = 1788, n = 1128)$;
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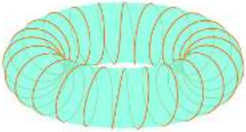
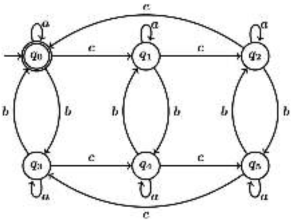
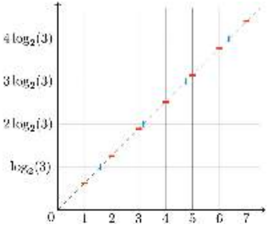


logic & automata theory



Diophantine geometry



Normality and Disjunctivity

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$\sqrt{2} = 1.41421356237309504880168872420969807856967187537694807317667973799$0
7324784621070388503875343276415727350138462309122970249248360558507372126
4412149709993583141322266592750559275579995050115278206057147010955997160
5970274534596862014728517418640889198609552329230484308714321450839762603
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7202264854470158588016207584749226572260020855844665214583988939443709265
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6912485905218100445984215059112024944134172853147810580360337107730918286
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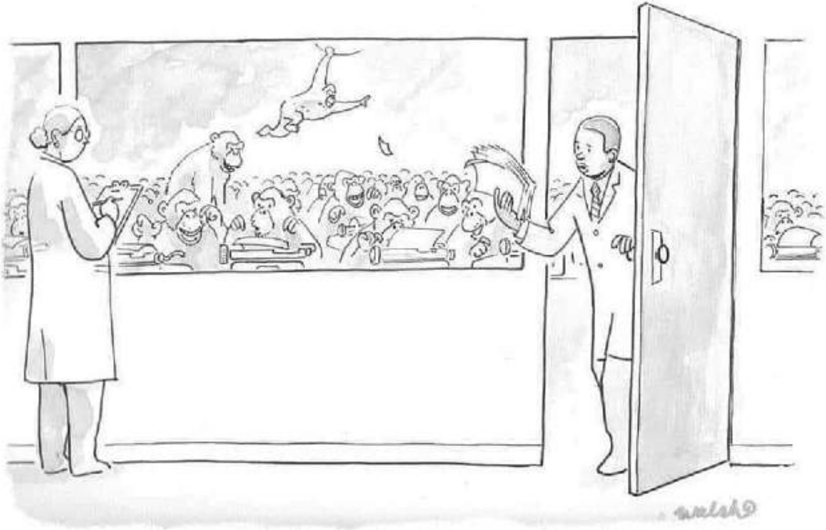
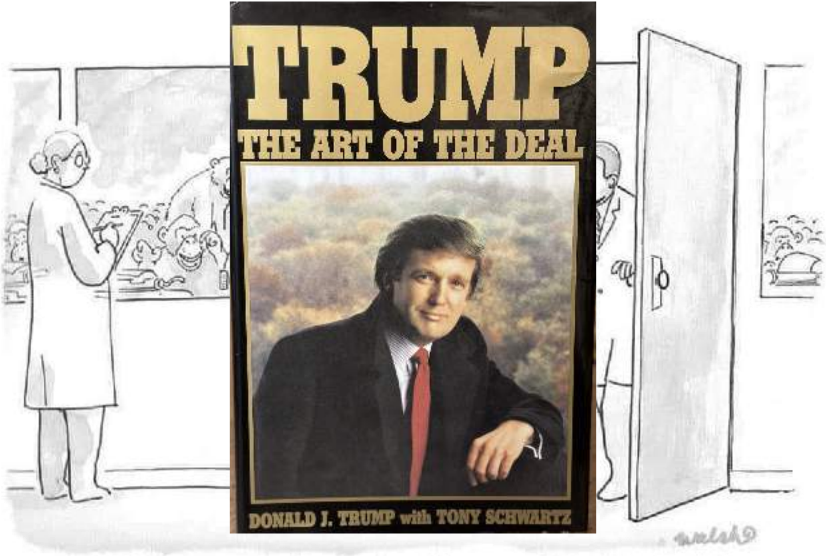


"No Shakespeare yet, but here's another copy of 'Art of the Deal'."



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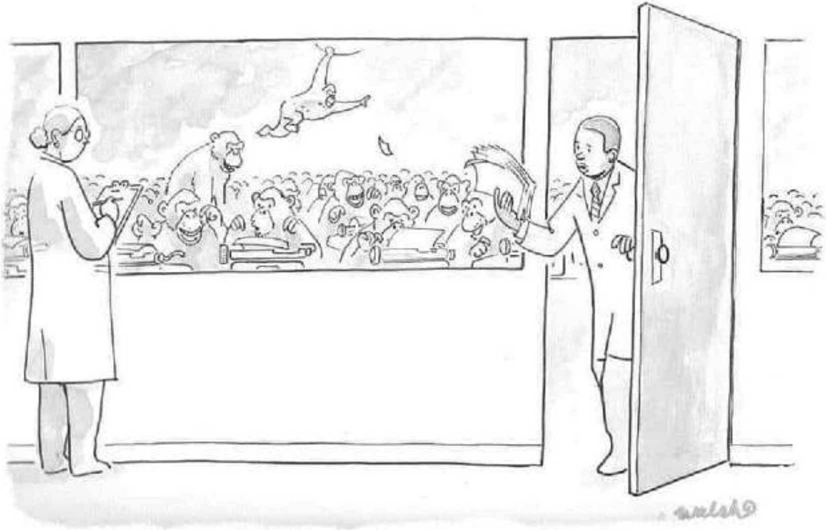

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Back to the Elgot-Rabin paper . . .

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- $\langle \mathbb{N}; <, \mathit{SQUARES} \rangle$
- $\langle \mathbb{N}; <, \mathit{CUBES} \rangle$
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Which predicates can one combine?

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Are there finitely many m and n such that

$$|2^m - 3^n| \leq 100\,?$$

If so, enumerate them.

There are 26 pairs in total;
the last one is $(m = 8, n = 5)$,
with $|2^8 - 3^5| = 13$

Are there infinitely many m and n such that

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and

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Yes, there are infinitely many.
The first pair is
$(m = 1788, n = 1128)$;
3^{1128} has 539 digits!

Are there finitely many n such that the number of perfect squares between 2^n and 2^{n+1} is even, *and* the number of perfect squares between 2^{n+1} and 2^{n+2} is divisible by 3?

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This is open! However:

- If $\sqrt{2}$ is disjunctive in binary, then there are infinitely many such n;
- If certain specific strings only occur *finitely often* in the binary expansion of $\sqrt{2}$, then there are only finitely many such n

Fragments of Hilbert's Program: More open problems

Open Problems

Is the MSO theory of the following structures decidable?

- $\langle \mathbb{N}; <, \mathit{SQUARES}, \mathit{CUBES} \rangle$
- $\langle \mathbb{N}; <, \mathit{SQUARES}, \mathit{FACT} \rangle$
- $\langle \mathbb{N}; <, \mathit{SQUARES}, \mathit{FIB} \rangle$
- $\langle \mathbb{N}; <, 2^{\mathbb{N}}, \mathit{FACT} \rangle$
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The *Brocard-Ramanujan Problem:* Find all integers m and n such that

$$n! + 1 = m^2$$

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- At the very least, would imply *effective* lower bounds on sums of S-units

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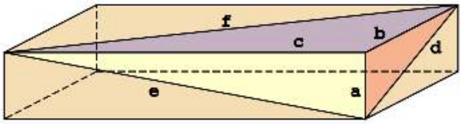

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Suppose that $\langle s_1, s_2, \ldots, s_M \rangle$ is an increasing sequence of *consecutive* perfect squares. Then for all $1 \leq n \leq M - 2$,

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- Note that this fragment is strong enough to express Büchi's Conjecture!

Outlook

Hilbert's legacy continues to be a rich source of inspiration!

Thriving research endeavour at the confluence of computer science and mathematics