Deciding the algebraic nature of D-finite power series

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Recurrence, transcendence & Diophantine approximation

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Goals, examples, motivation

Algebraic and transcendental power series

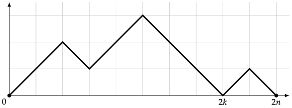
▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *algebraic* if it is a root of some algebraic equation $P(t, f(t)) = 0$, where $P \in \mathbb{Q}[x, y] \setminus \{0\}$.

Otherwise, f is called *transcendental*.

▷ Examples:

- polynomials in $\mathbb{Q}[t]$
- rational functions R in $\mathbb{Q}(t)$ with no pole at $t = 0$
- all powers R^α for $\alpha \in \mathbb{Q}$ and $R(0) = 1$
- sums and products of algebraic power series are algebraic
- the GF $\sum_{n\geq 0} C_n t^n$ of Dyck walks in $\mathbb{N}^2$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$



▷ Def extends to Laurent series $f \in \mathbb{Q}((t))$ and Puiseux series $f \in \overline{\mathbb{Q}}((t^{1/\star}))$

D-finite power series

▷ Definition: A power series f in $\mathbb{Q}[[t]]$ is called *D-finite* (*differentially finite*) if it is a solution of some LDE (i.e., *linear* ODE)

$$c_r(t)f^{(r)}(t) + \cdots + c_0(t)f(t) = 0$$

for some $c_i \in \mathbb{Q}(t)$, with c_r nonzero. (r is called the *order* of this LDE.)

Europ. J. Combinatorics (1980) **1**, 175–188

Differentiably Finite Power Series

R. P. Stanley*

A formal power series $\sum f(n)x^n$ is said to be differentiably finite if it satisfies a linear differential equation with polynomial coefficients. Such power series arise in a wide variety of problems in enumerative combinatorics. The basic properties of such series of significance to combinatorics are surveyed. Some reciprocity theorems are proved which link two such series together. A number of examples, applications and open problems are discussed.

1. Introduction

Recently there has been interest [2], [3], [16] in the problem of computing quickly the coefficients of a power series $F(x) = \sum_{n\geqslant 0} f(n)x^n$, where say $F(x)$ is defined by a functional equation or as a function of other power series. If the coefficients $f(n)$ have a combinatorial meaning, then a fast algorithm for computing $f(n)$ would also be of combinatorial interest. Here we consider a class of power series, which we call *differentiably finite* (or D-finite, for short), whose coefficients can be quickly computed in a simple way. We consider various operations on power series which preserve the property of being D-finite, and give examples of operations which don't preserve this property. We mention some classes of power series for which it seems quite difficult to decide whether they are D-finite. Everything we say can be extended routinely from power series to Laurent series having finitely many terms with negative exponents, though for simplicity we will restrict ourselves to power series. Moreover, we will consider only complex coefficients, though virtually all of what we do is valid over any field of characteristic zero (and much is valid over any field).

The class of D-finite power series has been subject to extensive investigation, particularly within the theory of differential equations. However, a systematic exposition of their properties from a combinatorial point of view seems not to have been given before. Many of our results can therefore be found scattered throughout the literature, so this paper should be regarded as about 75% expository. To simplify and unify the concepts and proofs we have used the terminology and elementary theory of linear algebra, though all explicit dependence on linear algebra could be avoided without great difficulty.

Let us now turn to the basic definition of this paper. First note that the field $\mathbb{C}((x))$ of all formal Laurent series over $\mathbb{C}$ of the form $\sum_{n\geqslant n_0} f(n)x^n$ for some $n_0 \in \mathbb{Z}$ contains the field $\mathbb{C}(x)$ of rational functions of x, and $\mathbb{C}((x))$ has the structure of a vector space over $\mathbb{C}(x)$.

Definition 1.1. A formal power series $y \in \mathbb{C}[[x]]$ is said to be *differentiably finite* (or D-finite) if y together with all its derivatives $y^{(n)} = d^n y/dx^n$, $n \geqslant 1$, span a finite-dimensional subspace of $\mathbb{C}((x))$, regarded as a vector space over the field $\mathbb{C}(x)$.

Theorem 1.2. *The following three conditions on a formal power series $y \in \mathbb{C}[[x]]$ are equivalent.*

(i) *y is D-finite.*

(ii) *There exist finitely many polynomials $q_0(x), \ldots, q_k(x)$, not all 0, and a polynomial $q(x)$, such that*

$$q_k(x)y^{(k)} + \cdots + q_1(x)y' + q_0(x)y = q(x). \quad (1)$$

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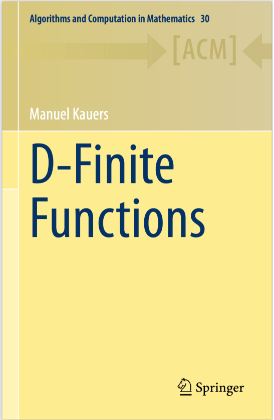

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$$c_r(t)f^{(r)}(t) + \cdots + c_0(t)f(t) = 0$$

for some $c_i \in \mathbb{Q}(t)$, with c_r nonzero. (r is called the *order* of this LDE.)

▷ Examples:

- $\exp(t) := \sum_{n\geq 0} t^n/n!$, solution of $f'(t) = f(t)$
- $\log(1-t) := -\sum_{n\geq 1} t^n/n$, solution of $(t-1)f''(t) + f'(t) = 0$
- $\sqrt[N]{R(t)}$ for $R \in \mathbb{Q}(t)$, solution of $f'(t)/f(t) = \frac{1}{N}R'(t)/R(t)$
- any algebraic power series is D-finite ("*Abel's theorem*")
- $\arctan(t)$, solution of $(t^2+1)f''(t) + 2tf'(t) = 0$, but not $\tan(t)$
- sums and products of D-finite are D-finite

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▷ Simple but important property: $\sum_{n\geq 0} a_n t^n$ is D-finite *if and only if* $(a_n)_{n\geq 0}$ is *P-finite* (i.e., it satisfies a linear recurrence with coefficients in $\mathbb{Q}[n]$).

Main question today: *How to decide if a D-finite power series is algebraic?*

In contrast with the "hard" theory of arithmetic transcendence, it is usually "easy" to establish transcendence of functions.

[Flajolet, Sedgewick, 2009]

Goal: Given a D-finite $f \in \mathbb{Q}[[t]]$, by a linear differential equation and enough initial terms, determine its *algebraicity* or *transcendence*.

▷ Example: What is the nature of $f(t) = 1 + 3t + 18t^2 + 105t^3 + \cdots$ such that

$$t^2(1+t)(1-2t)(1+4t)(1-8t)f'''(t) + t\left(576t^4 + 200t^3 - 252t^2 - 33t + 5\right)f''(t)$$
$$+4\left(288t^4 + 22t^3 - 117t^2 - 12t + 1\right)f'(t) + 12\left(32t^3 - 6t^2 - 12t - 1\right)f(t) = 0?$$

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Equivalent goal: Given a P-finite sequence of rational numbers $(a_n)_{n\geq 0}$ by a linear recurrence and enough initial terms, determine the *algebraicity* or the *transcendence* of its generating function $\sum_{n\geq 0} a_n t^n$.

▷ Example: What is the nature of $f(t) = \sum_{n\geq 0} a_n t^n$, where $(a_n)_{n\geq 0}$ is defined by $a_0 = 1, a_1 = 3, a_2 = 18, a_3 = 105$ and

$$(n+4)(n+5)^2 a_{n+4} - (n+4)\left(5n^2+43n+96\right) a_{n+3} - 6(5n+22)(n+4)(n+3) a_{n+2}$$
$$+8(n+2)\left(5n^2+15n+1\right) a_{n+1} + 64(n+3)(n+2)(n+1) a_n = 0?$$

▷ NB: Integrality and algebraicity are related; deciding integrality is harder!

Motivations

- Number theory: a first step towards proving the transcendence of a complex number is proving that some power series is transcendental

- Combinatorics: the nature of generating functions may reveal strong underlying structures

- Computer science: are algebraic power series (intrinsically) easier to manipulate?

Stanley's problem

Design an algorithm suitable for computer implementations which decides if a D-finite power series —given by a linear differential equation with polynomial coefficients and initial conditions— is algebraic, or not.

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$$f(t) = -t - \frac{t^2}{2} - \frac{t^3}{3} - \frac{t^4}{4} - \frac{t^5}{5} - \frac{t^6}{6} - \cdots$$

is D-finite and can be represented by the second-order LDE

$$\left((t-1)\partial_t^2 + \partial_t\right)(f) = 0, \quad f(0) = 0, f'(0) = -1.$$

▷ The algorithm should recognize (from this data) that f is *transcendental*.

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▷ The algorithm should recognize (from this data) that f is *transcendental*.

▷ The same algorithm should recognize the *algebraicity* of g such that

$$\left((t-1)\partial_t^2 + \partial_t\right)(g) = 0, \quad g(0) = -1, g'(0) = 0.$$

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▷ Notation: For a D-finite series f, we write $\mathscr{L}_f^{\min}$ for the least-order, monic, linear differential operator in $\mathbb{Q}(t)\langle\partial_t\rangle$ that cancels f.

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▷ Difficulty: $\mathscr{L}_f^{\min}$ might not be irreducible. E.g., $\mathscr{L}_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.

Related problems

$$\mathscr{L}(y(t)) := c_r(t)y^{(r)}(t) + \cdots + c_0(t)y(t) = 0$$

(S) *Stanley's problem*: Decide if a given solution f of $\mathscr{L}(y) = 0$ is algebraic

(F) *Fuchs' problem*: Decide if all solutions of $\mathscr{L}(y) = 0$ are algebraic

(L) *Liouville's problem*: Decide if $\mathscr{L}(y) = 0$ has at least one algebraic solution ($\neq 0$)

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Today: how to solve **(S), (F)** and **(L)** for arbitrary $\mathscr{L}$

Starting remarks, transcendence criteria, hypergeometric case

Starting remarks on **(F)**, **(L)** and **(S)**

▷ The minimal polynomial can have arbitrarily large size (degrees) w.r.t. the size (order/degree) of the differential equation:

solution of $N(t-1)f'(t) - f(t) = 0,\ f(0) = 1$ satisfies $f^N = 1 - t$

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▷ No characterization for coefficient sequences of algebraic power series

- larger class: *D-finite functions* $\Longleftrightarrow$ *P-finite sequences*
- smaller class: *rational functions* $\Longleftrightarrow$ *C-finite sequences*
- *diagonals* $\overset{\text{Christol's}}{\underset{\text{conjecture}}{\Longleftrightarrow}}$ *P-finite, almost integer, seq. with geometric growth*

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▷ Many tools: geometry (Schwarz, Klein), invariant theory (Fuchs, Gordan), group theory (Jordan, Painlevé), diff. Galois theory (Vessiot, Singer, Hrushovski), algebraic geometry (Grothendieck, Katz), number theory (Chudnovsky, André)

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- Arithmetic properties
  - f is globally bounded: $\exists C \in \mathbb{N}^*$ with $a_n C^n \in \mathbb{Z}$ for $n \geq 1$ [Heine, 1854]

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  In particular, *denominators of a_n's have finitely many prime divisors* [Eisenstein, 1852]
- Analytic properties$^{(\star)}$
  - $(a_n)_n$ has "nice" asymptotics [Puiseux, 1850; Darboux, 1878; Flajolet, 1987]

  Typically, $a_n \sim \kappa \, \rho^n \, n^\alpha$ with $\alpha \in \mathbb{Q} \setminus \mathbb{Z}_{<0}$ and $\rho \in \overline{\mathbb{Q}}$ and $\kappa \cdot \underbrace{\Gamma(\alpha+1)}_{:=\int_0^\infty t^\alpha e^{-t}\,\mathrm{d}t} \in \overline{\mathbb{Q}}$

$^{(\star)}$ "It is usually 'easy' to establish transcendence of functions, by exhibiting a local expansion that contradicts the Newton–Puiseux Theorem" [Flajolet, Sedgewick, 2009]

For $f = \sum_n a_n t^n \in \mathbb{Q}[[t]]$, if one of the following holds

- f is not D-finite

$$\underbrace{\sum_n p_n t^n = \prod_{n \geq 1} \frac{1}{1 - t^n}}_{1+t+2t^2+3t^3+5t^4+7t^5+11t^6+\cdots}$$

- f has infinitely many primes in the denominators

$$\sum_{n \geq 1} \frac{1}{n} t^n$$

- $(a_n)_n$ has incompatible asymptotics

$$\underbrace{\sum_{n \geq 0} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n}_{1+5t+73t^2+1445t^3+33001t^4+\cdots} \quad (\dagger)$$

then f is transcendental

(†) $a_n \sim \frac{(1+\sqrt{2})^{4n+2}}{2^{9/4}\pi^{3/2}n^{3/2}}$ [Cohen, 1978] and $\frac{\Gamma(-1/2)}{\pi^{3/2}} = -\frac{2}{\pi} \notin \overline{\mathbb{Q}}$ [von Lindemann, 1882]

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▷ None of these transcendence criteria is an equivalence!

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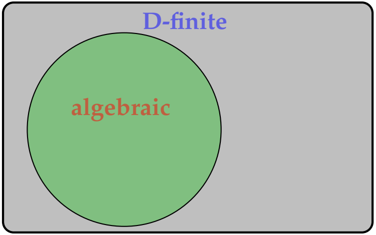

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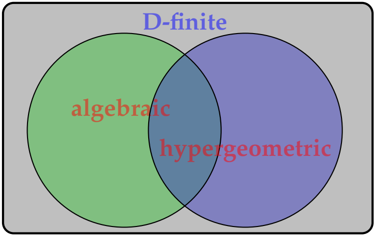
▷ *algebraic* if $P(t, f(t)) = 0$ for some $P(x, y) \in \mathbb{Z}[x, y] \setminus \{0\}$

▷ *D-finite* if $c_r(t) f^{(r)}(t) + \cdots + c_0(t) f(t) = 0$ for some $c_i \in \mathbb{Z}[t]$, not all zero



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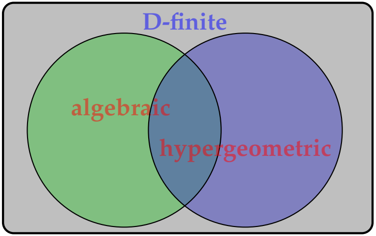
▷ *hypergeometric* if $\frac{a_{n+1}}{a_n} \in \mathbb{Q}(n)$. E.g., $\ln(1-t)$; $\frac{\arcsin(\sqrt{t})}{\sqrt{t}}$; $(1-t)^{\alpha}$, $\alpha \in \mathbb{Q}$

Hypergeometric case



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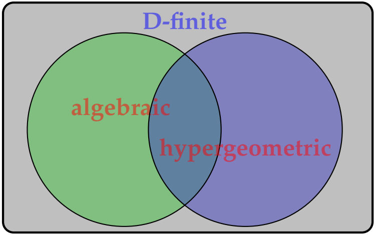

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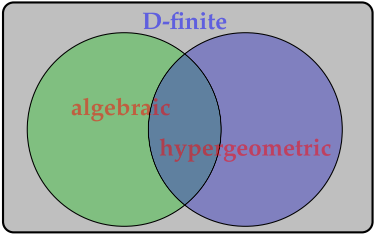

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Theorem [Schwarz 1873; Landau 1904, 1911; Stridsberg 1911; Errera 1913; Katz 1972; Christol 1986; Beukers, Heckman 1989; Katz 1990; Fürnsinn, Yurkevich 2024]

Full characterization of { *hypergeom* } ∩ { *algebraic* } + *algorithm* (!)

New results, examples

Main new results [B., Salvy, Singer, 2025]

- A *decidability result*: two theoretical/impractical algorithms for finding the algebraic or transcendental nature of D-finite power series in $\mathbb{Q}[[t]]$

- An incomplete but *practical transcendence test* D-finite series in $\mathbb{Q}[[t]]$:

  - always correct when it returns "transcendental"

  - may fail when it returns "algebraic"

  - always correct on differential equations with additional arithmetic properties (modulo a conjecture by Christol and André), e.g. if input encodes the diagonal of a multivariate rational function

- An *efficient implementation* `istranscendental` in `gfun` (`Maple`)

▷ Open: design transcendence tests that are both *complete* and *efficient*

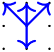
Three examples

(A) Apéry's power series [Apéry, 1978] (used in his proof of $\zeta(3) \notin \mathbb{Q}$)

$$\sum_n \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 t^n = 1 + 5\,t + 73\,t^2 + 1445\,t^3 + 33001\,t^4 + \cdots$$

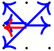
(B) GF of trident walks in the quarter plane

$$\sum_n a_n t^n = 1 + 2\,t + 7\,t^2 + 23\,t^3 + 84\,t^4 + 301\,t^5 + 1127\,t^6 + \cdots ,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

(C) GF of a quadrant model with repeated steps

$$\sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + 520\,t^6 + \cdots ,$$

where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

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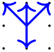
where $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

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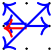
where $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

Question: *What is the nature of these three power series?*

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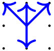
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ starting at $(0,0)\Big\}$

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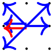
where $a_n = \#\Big\{$  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$

Answer: *Our implementation proves that they are all transcendental!*

Some timings: Green functions of the face-centered cubic lattice

dim d	order	degree	transc. (sec.)	factor. (sec.)
3	3	5	2.2	0.9
4	4	10	0.1	0.2
5	6	17	0.6	1.3
6	8	43	4.2	18.3
7	11	68	24.0	265.
8	14	126	174.9	4706.
9	18	169	771.6	$>$10000
10	22	300	8817.1	

$$G_d(t) := \frac{1}{\pi^d} \int_0^\pi \cdots \int_0^\pi \frac{d\theta_1 \cdots d\theta_d}{1 - t\binom{d}{2}^{-1} \sum_{1 \le i < j \le d} \cos\theta_i \cos\theta_j}$$

▷ 'transc' = time taken by `istranscendental` to prove transcendence of $G_d(t)$

▷ 'factor' = the time taken by factoring code of [Chyzak, Goyer, Mezzarobba 2022]

More examples: diagonals

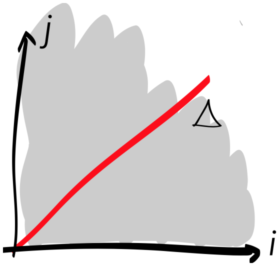
Definition

If F is a multivariate power series

$$F = \sum_{i_1,\dots,i_n \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

its diagonal is the univariate power series

$$\mathrm{Diag}(F) \stackrel{def}{=} \sum_i a_{i,\dots,i} t^i.$$



Theorem (Pólya, 1922)

Diagonals of series in $\mathbb{Q}(x, y)$ are algebraic.

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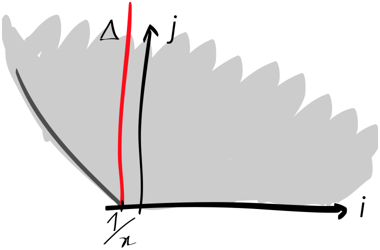

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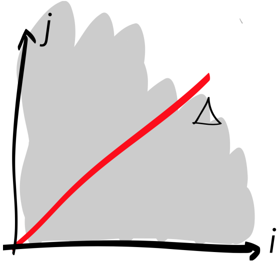

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Theorem (Christol, 1985)

Diagonals of power series in $\mathbb{Q}(x_1, \ldots, x_n)$ are D-finite.

- ▷ If $F = \frac{1}{(1-x-yz-z^2)(1-x-xy)}$, then $\underbrace{\mathrm{Diag}(F)}_{1+3t+11t^2+47t^3+211t^4+\cdots}$ is *algebraic*
- ▷ If $F = \frac{1}{(1-x-y-z^2)(1-x-xy)}$, then $\underbrace{\mathrm{Diag}(F)}_{1+56t^2+6391t^4+\cdots}$ is *transcendental*

Singer's algorithm
and
Stanley's problem

Problem **(F)**: Decide if *all solutions* of a given LDE $\mathscr{L}$ of order r *are algebraic*

• Starting point [Jordan, 1878]: If so, then for some solution y of $\mathscr{L}$, $u = y'/y$ has alg. degree at most $(49r)^{r^2}$ and satisfies a Riccati equation of order $r-1$

Algorithm ($\mathscr{L}$ irreducible) [Painlevé, 1887], [Boulanger, 1898], [Singer, 1979]

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2. (Abel's problem) Given an algebraic u, decide whether $y'/y = u$ has an algebraic solution y — [Risch 1970], [Baldassarri & Dwork 1979]

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▷ [Singer, 2014; B., Salvy, Singer, 2025]: compute $\mathscr{L}^{\text{alg}}$, factor of $\mathscr{L}$ whose solution space is spanned by alg. solutions of $\mathscr{L}$ $\longrightarrow$ requires LDE factoring

Application to Stanley's problem

Problem **(S)**: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ and sufficiently many initial terms, is transcendental.

First algorithm for problem **(S)** [B., Salvy, Singer, 2025]

1. Compute $\mathscr{L}^{\text{alg}}$
2. Decide if $\mathscr{L}^{\text{alg}}$ annihilates f

▷ Benefit: Solves (in theory) problems **(S)**, **(F)**, **(L)**: *algebraicity is decidable*

▷ Drawbacks: Step 1 involves *impractical bounds* & *requires LDE factorization*

▷ LDE factorization is decidable
[Fabry, 1885], [Markov, 1891], [van Hoeij, 1997], [van der Hoeven, 2007], ...

▷ ... but possibly extremely costly: complexity $(N\mathcal{L})^{O(r^4)}$,
with $\mathcal{L} = \text{bitsize}(\mathscr{L})$ and $N = e^{(\mathcal{L}\cdot 2^r)^{o(2^r)}}$ [Grigoriev, 1990]

A practical method, based on Minimization

Practical method: the main idea [B., Salvy, Singer, 2025]

Problem **(S)**: Decide if a D-finite power series $f \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ and sufficiently many initial terms, is transcendental.

Recall: $\mathscr{L}_f^{\min}$:= least-order, monic, in $\mathbb{Q}(t)\langle\partial_t\rangle$, such that $\mathscr{L}_f^{\min}(f) = 0$

Key property: If f is algebraic, then $M := \mathscr{L}_f^{\min}$ has algebraic solutions only.
Proof: $M = QM^{\text{alg}}$, $M^{\text{alg}}(f) = 0$ and minimality imply $Q = 1$, so $M = M^{\text{alg}}$.

Corollary: If $\mathscr{L}_f^{\min}$ has a log singularity, then f is transcendental.

▷ Pros and cons: *Avoids factorization* of $\mathscr{L}$, but *requires computing* $\mathscr{L}_f^{min}$.

Ex. (A): Apéry's power series

Theorem (Apéry's power series is transcendental)

$$f(t) = \sum_n A_n t^n, \quad \text{where } A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2, \quad \text{is transcendental.}$$

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Proof:

1. Creative telescoping: [Zagier, 1979], [Zeilberger, 1990]

$$(n+1)^3 A_{n+1} + n^3 A_{n-1} = (2\,n+1)\,(17\,n^2 + 17\,n + 5) A_n, \quad A_0 = 1,\ A_1 = 5$$

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5. Conclusion: f is transcendental†

---

† f algebraic would imply a full basis of algebraic solutions for $\mathscr{L}_f^{\min}$.

Deciding algebraicity via minimization

Input: A D-finite $f(t) \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ plus initial terms
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

▷ Principle: **(S)** is reduced to **(F)** via minimization

Second algorithm for problem (S) [B., Salvy, Singer, 2025]

1. Compute $\mathscr{L}_f^{\min}$ [B., Rivoal, Salvy, 2024]
2. Decide if $\mathscr{L}_f^{\min}$ has only algebraic solutions; if so return A, else return T [Singer, 1979]

▷ Benefit: Solves (in theory) Stanley's problem **(S)**: *algebraicity is decidable*

▷ Drawback: Step 2 can be very costly in practice

Input: A D-finite $f(t) \in \mathbb{Q}[[t]]$, given by an LDE $\mathscr{L}(f) = 0$ plus initial terms
Output: T if $f(t)$ is transcendental, A if $f(t)$ is algebraic

Third algorithm for problem (S) [B., Salvy, Singer, 2025]

1. Compute $\mathscr{L}_f^{\min}$ [B., Rivoal, Salvy, 2024]
2. If $\mathscr{L}_f^{\min}$ has a logarithmic singularity, return T; otherwise return A

▷ This algorithm is always correct when it returns T

▷ *Conjecturally*, under the additional assumption that *f is globally bounded*[♢], it is also always correct[♣] when it returns A [Christol, 1986], [André, 1997]

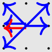
▷ *Efficient implementation* `istranscendental` in `gfun` (`Maple`)

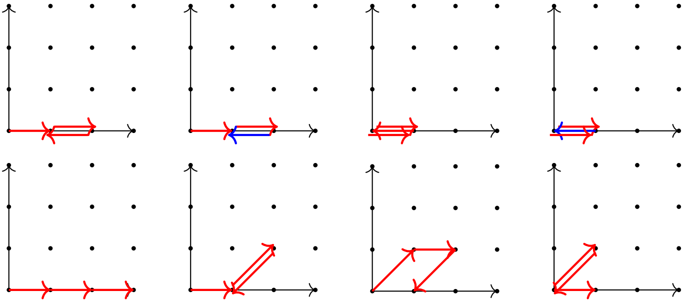
[♢] E.g. if f is given as GF of a binomial sum, or as the diagonal of a rational function

[♣] NB: not true without the global boundedness assumption, e.g. $f(t) = {}_2F_1\left(\begin{matrix}\frac{1}{6} & \frac{5}{6} \\ \frac{7}{6} & \end{matrix} \middle| \, t\right)$

Ex. (C): a difficult quadrant model with repeated steps

Theorem [B., Bousquet-Mélou, Kauers, Melczer, 2016]

Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.



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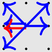
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

Proof:

1. Discover and certify a differential equation $\mathscr{L}$ for $f(t)$ of order 11 and degree 73 — high-tech Guess-and-Prove
2. If $\operatorname{ord}(\mathscr{L}_f^{\min}) \leq 10$, then $\deg_t(\mathscr{L}_f^{\min}) \leq 580$ — apparent singularities
3. Rule out this possibility — differential Hermite-Padé approximants
4. Thus, $\mathscr{L}_f^{\min} = \mathscr{L}$
5. $\mathscr{L}$ has a log singularity at $t = 0$, and so f is transcendental $\square$

▷ Computer-driven discovery and proof; no human proof yet

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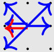
Let $a_n = \#$ {  – walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)$ }. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

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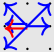
Let $a_n = \#\Big\{$  $-$ walks of length n in $\mathbb{N}^2$ from $(0,0)$ to $(\star,0)\Big\}$. Then $f(t) = \sum_n a_n t^n = 1 + t + 4\,t^2 + 8\,t^3 + 39\,t^4 + 98\,t^5 + \cdots$ is transcendental.

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▷ Computer-driven discovery and proof; no human proof yet
▷ All other criteria and algorithms fail or do not terminate
▷ `istranscendental` takes about 10 seconds to prove transcendence

Problem: Given a D-finite power series $f \in \mathbb{Q}[[t]]$ by a differential equation $\mathscr{L}(f) = 0$ and sufficiently many initial terms, compute $\mathscr{L}_f^{\min}$.

▷ Why isn't this easy? After all, it is just a differential analogue of:

Given an algebraic power series $f \in \mathbb{Q}[[t]]$
by an algebraic equation $P(t, f) = 0$ and sufficiently many initial terms,
compute its minimal polynomial P_f^{min}.

▷ $\mathscr{L}_f^{\min}$ is a (right) factor of $\mathscr{L}$, but contrary to the commutative case:

- $\mathscr{L}_f^{\min}$ might not be irreducible. E.g., $\mathscr{L}_{\ln(1-t)}^{\min} = \left(\partial_t + \frac{1}{t-1}\right)\partial_t$.
- factorization of diff. operators is not unique $\quad \partial_t^2 = (\partial_t + \frac{1}{t-c})(\partial_t - \frac{1}{t-c})$
- ...and it is difficult to compute
- $\deg_t \mathscr{L}_f^{\min} > \deg_t \mathscr{L}$, due to apparent singularities $\qquad (t\partial_t - N) \mid \partial_t^{N+1}$

▷ $\deg_t \mathscr{L}_f^{\min}$ can be bounded w.r.t. n and local data of $\mathscr{L}$ via Fuchs' relation

Summary

- Problems **(S)**, **(F)**, **(L)** on algebraicity of solutions of LDEs are decidable
- In practice, proving *transcendence is easier* than proving algebraicity (!)
- LDE minimization is a practical alternative for proving transcendence
  - ☺ $\longrightarrow$ allows to solve difficult problems from applications
  - ☺ $\longrightarrow$ also useful in other contexts (effective Siegel-Shidlovskii)
- Guess-and-Prove is a powerful method for proving algebraicity
  - ☺ $\longrightarrow$ robust: adapts to other functional equations
  - ☹ $\longrightarrow$ main limitation: output size!
- Brute-force / naive algorithms $\longrightarrow$ hopeless on "real-life" applications

Further questions

- How to decide in practice if *two (or more) D-finite* power series are algebraically (in)dependent?

- How to decide in practice if a *bivariate (or multivariate)* D-finite power series is algebraic or transcendental?

- Design *effective and efficient versions* of (proved cases of) the Grothendieck-Katz conjecture (e.g., Honda 1974, Chudnovsky^2 1985)
  $\longrightarrow$ work in progress by Fürnsinn and Pannier

- How to decide if a *P-recursive sequence* has (almost) integral terms?
  $\longrightarrow$ work in progress by B. and Matveeva

- How to decide in practice if a D-finite power series is algebraic, in very difficult cases where Guess-and-Prove does not terminate?
  $\longrightarrow$ work in progress by B., Weil, Yurkevich

Thanks for your attention!