Variations on Schanuel's Conjecture for elliptic and quasi-elliptic functions: the split case

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Schanuel Conjecture

Consider the exponential map of the multiplicative group $\mathbb{G}_m$

$$\begin{aligned} \exp_{\mathbb{G}_m} : \mathrm{Lie}\, \mathbb{G}_m = \mathbb{C} &\longrightarrow \mathbb{G}_m(\mathbb{C}) = \mathbb{C}^* \\ z &\longmapsto \mathrm{e}^z. \end{aligned}$$

In 1960 Schanuel proposed a conjecture, which is supposed to contain all the "reasonable" statements that can be made on the values of the exponential map:

Schanuel Conjecture

Let $t_1, \ldots, t_n$ be complex numbers, linearly independent over $\mathbb{Q}$. Then at least n of the numbers $t_1, \ldots, t_n, \mathrm{e}^{t_1}, \ldots, \mathrm{e}^{t_n}$ are algebraically independent over $\overline{\mathbb{Q}}$.

This conjecture is wide open, except for:

- the **Lindemann-Weierstrass Theorem** : if $t_1, \ldots, t_n$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then $\mathrm{e}^{t_1}, \ldots, \mathrm{e}^{t_n}$ are algebraically independent over $\overline{\mathbb{Q}}$,
- the **Hermite-Lindemann Theorem** : Schanuel conjecture with $n = 1$.

Our aim

Let G be an extension of an elliptic curve $\mathcal{E}$ defined over $\mathbb{C}$ by the multiplicative group $\mathbb{G}_m$:

$$0 \to \mathbb{G}_m \to G \to \mathcal{E} \to 0.$$

AIM: state conjectures *à la Schanuel* for the exponential map $\exp_G : \mathrm{Lie}\, G \to G$ of G

Until now we have done the split case: $G = \mathbb{G}_m \times \mathcal{E}$. Today we will speak about this case.

Our conjecture *à la Schanuel* will imply a conjecture *à la Lindemann-Weierstrass*

The split semi-elliptic exponential map

- $\mathcal{E}$ elliptic curve over $\mathbb{C}$,
- g_2, g_3 its invariants,
- $\Omega = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ its lattice,
- $\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right)$ the Weierstrass $\wp$-function,
- $\zeta(z) = \frac{1}{z} + \sum_{\substack{\omega\in\Omega \\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2}\right)$ the Weierstrass ζ-function,
- k the field of endomorphisms of $\mathcal{E}$

$$k := \mathrm{End}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{in the non–CM case,} \\ \mathbb{Q}(\tau) & \text{in the CM case, where } \tau := \omega_2/\omega_1. \end{cases}$$

The exponential map of the product $G = \mathbb{G}_m \times \mathcal{E}$ involves the exponential function and the Weierstrass $\wp$-function

$$\begin{aligned} \exp_{\mathbb{G}_m\times\mathcal{E}} : \mathrm{Lie}(\mathbb{G}_m \times \mathcal{E}) &\longrightarrow (\mathbb{G}_m \times \mathcal{E})(\mathbb{C}) \subset (\mathbb{G}_m \times \mathbb{P}^2)(\mathbb{C}) \\ (t, z) &\longmapsto \left(\mathrm{e}^t, [\wp(z) : \wp'(z) : 1]\right) \end{aligned}$$

Main Conjecture

Split Semi–Elliptic Schanuel Conjecture

Let $t_1, \ldots, t_s \in \mathbb{C}$ be $\mathbb{Q}$–linearly independent.
Let $p_1, \ldots, p_n \in \mathbb{C} \smallsetminus \Omega$ be k–linearly independent.
Then the transcendence degree of the field

$$K := \mathbb{Q}\big(t_1, \ldots, t_s, e^{t_1}, \ldots, e^{t_s}, g_2, g_3, p_1, \ldots, p_n, \wp(p_1), \ldots, \wp(p_n), \zeta(p_1), \ldots, \zeta(p_n)\big)$$

is at least $s + 2n$, unless $2\pi\mathrm{i}\mathbb{Q} \subset \mathbb{Q}t_1 + \cdots + \mathbb{Q}t_s$ and $\Omega \subset kp_1 + \cdots + kp_n$, in which case it is at least $s + 2n - 1$.

If $n = 0$ and there is no elliptic curve, we get Schanuel Conjecture. In this case $\Omega \cap (kp_1 + \cdots + kp_n) = \{0\}$.

If $s = 0$, we get the so called elliptic Schanuel Conjecture. In this case $2\pi\mathrm{i} \notin \mathbb{Q}t_1 + \cdots + \mathbb{Q}t_s$.

Lindemann-Weierstrass analogue

Split Semi–Elliptic LW Conjecture

Assume the algebraic invariants g_2, g_3 to be algebraic. If

- $t_1, \ldots, t_s$ are $\mathbb{Q}$–linearly independent algebraic numbers,
- $p_1, \ldots, p_n$ are k–linearly independent algebraic numbers,

then the $s + 2n$ numbers $\mathrm{e}^{t_1}, \ldots, \mathrm{e}^{t_s}, \wp(p_1), \ldots, \wp(p_n), \zeta(p_1), \ldots, \zeta(p_n)$ are algebraically independent.

Two cases of this Conjecture are known:

- the **Lindemann-Weierstrass Theorem** : if $t_1, \ldots, t_n$ are algebraic numbers that are linearly independent over $\mathbb{Q}$, then $\mathrm{e}^{t_1}, \ldots, \mathrm{e}^{t_n}$ are algebraically independent over $\overline{\mathbb{Q}}$,
- the **Philippon–Wüstholz Theorem** : the values of a Weierstrass $\wp$ function, with algebraic invariants and with complex multiplication, at k–linearly independent algebraic numbers, are algebraically independent.

Why we put Weierstrass ζ-function ?

In my Ph.D. thesis I proved that Schanuel Conjecture is **equivalent to** the Grothendieck-André Period Conjecture applied to the 1-motive

$$M = [u : \mathbb{Z} \to \mathbb{G}_m^s],\ u(1) = (\mathrm{e}^{t_1}, \dots, \mathrm{e}^{t_s}) \in \mathbb{G}_m^s(\mathbb{C})$$

We want to proceed in a manner compatible with Grothendieck-André:

In fact our Split Semi-Elliptic Conjecture is **equivalent to** the Grothendieck-André Period Conjecture applied to the 1-motive

$$M = [u : \mathbb{Z} \to \mathbb{G}_m^s \times \mathcal{E}^n],\ u(1) = (\mathrm{e}^{t_1}, \dots, \mathrm{e}^{t_s}, P_1, \dots, P_n) \in (\mathbb{G}_m^s \times \mathcal{E}^n)(\mathbb{C}),$$

where $P_i = [\wp(p_i) : \wp'(p_i) : 1]$ for $i = 1, \dots, n$.

Elliptic differentials and elliptic integrals

On the elliptic curve $\mathcal{E}$ (defined over $K = \mathbb{Q}(g_2, g_3)$) we have the following differential forms:

1. a differential of the first kind $\omega = \frac{dx}{y}$ and $\exp_{\mathcal{E}}^*(\omega) = dz$.
2. a differential of the second kind $\eta = -\frac{xdx}{y}$ and $\exp_{\mathcal{E}}^*(\eta) = -\wp(z)dz$

A basis of the K-vector space $\mathrm{H}^1_{\mathrm{dR}}(\mathcal{E})$ is $\{\omega, \eta\}$.

Let γ_1, γ_2 be a basis for the $\mathbb{Z}$-module $\mathrm{H}_1(\mathcal{E}(\mathbb{C}), \mathbb{Z})$. We have that

1. the elliptic integrals of the first kind $\int_{\gamma_1} \omega = \omega_1$ and $\int_{\gamma_2} \omega = \omega_2$ are the **periods** of the Weierstrass $\wp$-function:

$$\wp(z + \omega_i) = \wp(z) \qquad \text{for } i = 1, 2.$$

2. the elliptic integrals of the second kind $\int_{\gamma_1} \eta = \eta_1$ and $\int_{\gamma_2} \eta = \eta_2$ are the **quasi-periods** of the Weierstrass ζ-function:

$$\zeta(z + \omega_i) = \zeta(z) + \eta_i \qquad \text{for } i = 1, 2.$$

Grothendieck Theorem and Conjecture 1966

The integration is a perfect pairing between de Rham and Betti cohomologies of an elliptic curve:

$$\begin{array}{ccc} \mathrm{H}^1_{\mathsf{dR}}(\mathcal{E}) \otimes_K \mathbb{C} \times \mathrm{H}_1(\mathcal{E}(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} & \longrightarrow & \mathbb{C} \\ (\omega, \gamma) & \longmapsto & \int_\gamma \omega \ . \end{array}$$

The entries of the matrix $\begin{pmatrix} \omega_1 & \eta_1 \\ \omega_2 & \eta_2 \end{pmatrix}$ representing this isomorphism are called the **periods** of $\mathcal{E}$.

Grothendieck Period Conjecture by S. Lang

Any polynomial relation (with rational coefficients) between the periods of $\mathcal{E}$ should have a **geometrical origin**: more precisely, any algebraic cycle on $\mathcal{E}$ and on the products of $\mathcal{E}$ with itself, will give rise to a polynomial relation (with rational coefficients) among the periods of $\mathcal{E}$.

Period matrix

$M = [u : \mathbb{Z} \to \mathbb{G}_m^s \times \mathcal{E}^n]$, $u(1) = (\mathrm{e}^{t_j}, P_i)_{i,j}$, where $P_i = [\wp(p_i) : \wp'(p_i) : 1]$.
Choosing adequate bases we have

$$\begin{pmatrix} 1 & p_1 & \zeta(p_1) & \dots & \dots & p_n & \zeta(p_n) & t_1 & \dots & t_s \\ & \omega_1 & \eta_1 & & & & & & & \\ & \omega_2 & \eta_2 & & & 0 & & & & \\ & & & \ddots & & & & & 0 & \\ & & 0 & & \ddots & & & & & \\ & & & & & \omega_1 & \eta_1 & & & \\ 0 & & & & & \omega_2 & \eta_2 & & & \\ & & & & & & & & 2\pi\mathrm{i}\mathrm{Id}_{s\times s} & \end{pmatrix}$$

Remark : Because of the differential form of second kind η, the Weierstrass ζ-function appears here as a period ! $(\exp_{\mathcal{E}}^*(\eta) = -\wp(z)dz = d\zeta(z))$

Grothendieck-André Period Conjecture

André has proposed the following conjecture:

Grothendieck-André Period Conjecture

Let M be a (pure or mixed) motive defined over a sub-field K of $\mathbb{C}$, then

$$\mathrm{tr.deg.}_{\mathbb{Q}} K(\mathrm{periods}(M)) \geq \dim \mathcal{G}_{\mathrm{mot}}(M)$$

Remarks:

1. in our cases $\mathcal{G}_{\mathrm{mot}}(M) = \mathrm{MT}(M)$
2. if $K \subseteq \overline{\mathbb{Q}}$, we have an equality
3. the only known case of this conjecture is **Chudnovsky Theorem**: let $\mathcal{E}$ be an elliptic curve defined over $\overline{\mathbb{Q}}$ with complex multiplication (i.e. $\mathrm{End}(\mathcal{E}) \supsetneq \mathbb{Z}$) then
$$\mathrm{tr.deg.}_{\mathbb{Q}}\, \mathbb{Q}(\omega_1, \omega_2, \eta_1, \eta_2) = 2.$$

Motivic Galois group or Mumfort-Tate group

For elliptic curve we know that

$$\mathcal{G}_{\mathrm{mot}}(\mathcal{E}) = \begin{cases} \mathbb{G}_m^2 \ \ (\dim 2) & \text{CM case (i.e. } \mathrm{End}(\mathcal{E}) \supset \mathbb{Z}), \\ \mathrm{GL}_2(\mathbb{Q}) \ \ (\dim 4) & \text{non CM case (i.e. } \mathrm{End}(\mathcal{E}) = \mathbb{Z}) \end{cases} .$$

For our $M = [u : \mathbb{Z} \to \mathbb{G}_m^s \times \mathcal{E}^n]$, $u(1) = (\mathrm{e}^{t_1}, \dots, \mathrm{e}^{t_s}, P_1, \dots, P_n) \in (\mathbb{G}_m^s \times \mathcal{E}^n)(\mathbb{C})$, we have the short exact sequence

$$0 \to \mathrm{UR}(M) \to \mathcal{G}_{\mathrm{mot}}(M) \to \mathcal{G}_{\mathrm{mot}}(\mathcal{E}) \to 0$$

where $\mathrm{UR}(M)$ is the unipotent radical of $\mathcal{G}_{\mathrm{mot}}(M)$ and $\mathcal{G}_{\mathrm{mot}}(\mathcal{E})$ is its largest reductive quotient.
Hence,

$$\dim \mathcal{G}_{\mathrm{mot}}(M) = \dim \mathcal{G}_{\mathrm{mot}}(\mathcal{E}) + \dim \mathrm{UR}(M)$$

Conclusion

Recalling that $k = \mathrm{End}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $M = [u : \mathbb{Z} \to \mathbb{G}_m^s \times \mathcal{E}^n]$, $u(1) = (\mathrm{e}^{t_j}, P_i)_{i,j}$, where $P_i = [\wp(p_i) : \wp'(p_i) : 1]$

$$\dim \mathcal{G}_{\mathrm{mot}}(M) = \begin{cases} 2\ (\mathrm{CM}), \\ 4\ (\mathrm{non\ CM}) \end{cases} + 2 \dim_k kp_i/(k\omega_1 + k\omega_2) + \dim_{\mathbb{Q}} \mathbb{Q}t_j/\mathbb{Q}2\pi\mathrm{i}$$

Remark : If the p_i are k-linearly dependent and/or if t_j are $\mathbb{Q}$-linearly dependent, $\dim \mathcal{G}_{\mathrm{mot}}(M)$ decreases !

polynomials VS endomophisms

an increase in the numbers of endomorphisms and/or an increase in the linear dependence between the p_i and the t_j $\Leftrightarrow$

an decrease in the dimension of $\mathcal{G}_{\mathrm{mot}}(M)$ $\Leftrightarrow$

an increase in the numbers of polynomial relations between periods of M.

Schanuel VS Grothendieck-André

Grothendieck-André Periods Conjecture applied to $M = [u : \mathbb{Z} \to \mathbb{G}_m^s \times \mathcal{E}^n],\ u(1) = (\mathrm{e}^{t_j}, P_i)$

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(g_2, g_3, \wp(p_i), \mathrm{e}^{t_j})(\omega_1, \eta_1, \omega_2, \eta_2, 2\pi\mathrm{i}, p_i, \zeta(p_i), t_j)_{i,j} \geq$$

$$\begin{cases} 2 \text{ (CM)} \\ 4 \text{ (non CM)} \end{cases} + 2\dim_k kp_i/(k\omega_1 + k\omega_2) + \dim_{\mathbb{Q}} \mathbb{Q}t_j/\mathbb{Q}2\pi\mathrm{i}$$

$$\Longleftrightarrow$$

Split Semi–Elliptic Schanuel Conjecture

Let $t_1, \dots, t_s \in \mathbb{C}$ be $\mathbb{Q}$–linearly independent.
Let $p_1, \dots, p_n \in \mathbb{C} \smallsetminus \Omega$ be k–linearly independent.
Then the transcendence degree of the field

$$K := \mathbb{Q}\big(t_1, \dots, t_s, e^{t_1}, \dots, e^{t_s}, g_2, g_3, p_1, \dots, p_n, \wp(p_1), \dots, \wp(p_n), \zeta(p_1), \dots, \zeta(p_n)\big)$$

is at least $s + 2n$, unless $2\pi\mathrm{i}\mathbb{Q} \subset \mathbb{Q}t_1 + \cdots + \mathbb{Q}t_s$ and $\Omega \subset kp_1 + \cdots + kp_n$, in which case it is at least $s + 2n - 1$.

Needed results

Division and torsion points by $m \in \mathbb{Z}, m \geqslant 1$

1. Let $p \in \mathbb{C} \setminus \Omega$. Then $\wp(p/m)$ and $\zeta(p/m) - \zeta(p)/m$ are algebraic over the field $\mathbb{Q}(g_2, g_3, \wp(p))$.
2. Let $\omega \in \Omega$. Assume $\omega/m \notin \Omega$. Then $\wp(\omega/m)$ and $\zeta(\omega/m) - \eta(\omega)/m$ are algebraic over the field $\mathbb{Q}(g_2, g_3)$.

Action of endomorphisms - $k = \mathrm{End}(\mathcal{E}) \otimes_{\mathbb{Z}} \mathbb{Q}$

Let $\alpha \in k$. Write $\alpha = r_1 + r_2\tau$ where $r_1, r_2 \in \mathbb{Q}$, not both zero. Let $m \in \mathbb{Z}$ be the least positive integer such that mr_1 and mr_2/C are integers (τ is a root of the polynomial $A + BX + CX^2 = 0$). Then

(1) The function $\wp(\alpha z)$ belongs to $k(g_2, g_3, \wp(z/m))$.

(2) The function Ξ_{r_1,r_2} defined by

$$\zeta(\alpha z) = \left(r_1 + \frac{A}{C\tau} r_2\right)\zeta(z) - \frac{\kappa r_2}{C} z + \Xi_{r_1,r_2}(z)$$

belongs to $k(g_2, g_3, \wp(z/m), \wp'(z/m))$ (here $\kappa \in \overline{\mathbb{Q}(g_2, g_3)}$).

Summary

Recall that

$$\dim \mathcal{G}_{\mathrm{mot}}(M) = \begin{cases} 2 \text{ (CM)}, \\ 4 \text{ (non CM)} \end{cases} + 2 \dim_k kp_i/(k\omega_1 + k\omega_2) + \dim_{\mathbb{Q}} \mathbb{Q}t_j/\mathbb{Q}2\pi \mathrm{i}.$$

Corollary

Let $\omega \in \Omega$, let $p_1, \ldots, p_{n+1}$ in $\mathbb{C} \setminus \Omega$ and let $b_0, b_1, \ldots, b_n$ in k. Assume

$$p_{n+1} = b_0\omega + b_1 p_1 + \cdots + b_n p_n.$$

Then $\wp(p_{n+1})$ and $\zeta(p_{n+1})$ are algebraic over the field

$$\mathbb{Q}(g_2, g_3, \omega_1, \omega_2, \eta_1, \eta_2, p_1, \ldots, p_n, \wp(p_1), \ldots, \wp(p_n), \zeta(p_1), \ldots, \zeta(p_n)).$$

Consequences

We expect that our Split Semi–Elliptic Schanuel Conjecture contains all "reasonable" statements that can be made on the values of the exponential function and the Weierstrass $\wp$ and ζ functions.

We checked that it implies some Schneider's theorems as

Schneider's theorem

Let $p \in \mathbb{C} \smallsetminus \Omega$. Assume g_2, g_3 and $\wp(p)$ are algebraic. Then p is transcendental. Further, let $\alpha \in \overline{\mathbb{Q}} \smallsetminus k$. Then $\alpha p \notin \Omega$ and $\wp(\alpha p)$ is transcendental.

.... and a

Waldschmidt's conjecture

Assume that g_2 and g_3 are algebraic. Let $p \in \mathbb{C} \smallsetminus \Omega$. Assume that $\zeta(p)$ is algebraic. Let α be an algebraic number different from $0, 1, -1$. Assume also that $\alpha^4 \neq 1$ if $g_3 = 0$ and $\alpha^6 \neq 1$ if $g_2 = 0$. Then $\alpha p \notin \Omega$ and $\zeta(\alpha p)$ is transcendental.

THE END !