Recurrence, transcendence, and Diophantine approximation
Leiden, 17/07/25

UNIVERSITÉ PARIS-EST CRÉTEIL VAL DE MARNE

On the $P(t)$–adic Littlewood Conjecture

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Outline

Littlewood's Conjecture

- Notation :

$$\langle \cdot \rangle \; = \; \mathrm{dist}\,(\,\cdot\,,\mathbb{Z})\,.$$

Conjecture (Littlewood, c. 1930)

For all $(\alpha, \beta) \in \mathbb{R}^2$,

$$\inf_{q \geq 1} q \cdot \langle q\alpha \rangle \cdot \langle q\beta \rangle \; = \; 0.$$

Figure – John Edensor Littlewood (1885 – 1977)

The p-adic Littlewood Conjecture

Recall : According to the Littlewood conjecture, $\inf_{q\geq 1} q \cdot \langle q\alpha \rangle \cdot \langle q\beta \rangle = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$.

Conjecture (De Mathan & Teulié, 2004)

For all $\alpha \in \mathbb{R}$ and all prime number p,

$$\inf_{q\geq 1} q \cdot |q|_p \cdot \langle q\alpha \rangle = 0,$$

where $|q|_p = p^{-k}$ if $q = p^k n$ with $p^{k+1} \nmid n$. Equivalently,

$$\inf_{n\geq 1, k\geq 0} n \cdot \langle np^k \alpha \rangle = 0.$$

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Towards an analogous theory over function fields

- Let α be a real number (e.g., $\alpha = 567.789\,908\,782\ \ldots$) :

$$\alpha \;=\; \underbrace{b_{-n_0}\cdots b_{-1}b_0}_{\text{integer part}} . \underbrace{b_1 b_2 b_3\cdots}_{\text{fractional part}} \;=\; \underbrace{\sum_{k=0}^{n_0} b_{-k}10^k}_{\text{integer part}} + \underbrace{\sum_{k=1}^{\infty}\frac{b_k}{10^k}}_{\text{fractional part}} .$$

- Let $\mathbb{K}$ be a finite field and let $A(t) \in \mathbb{K}\left(\left(t^{-1}\right)\right)$ be a Laurent series :

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Real case	Functional case
$\mathbb{Z}$	$\mathbb{K}[t]$
prime p	$P(t) \in \mathbb{K}[t]$ irreducible
$\mathbb{Q}$	$\mathbb{K}(t)$
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Littlewood conjectures over function fields

Definition (Norm of a Laurent series)

Given a Laurent series

$$A(t) \;=\; b_{-n_0}t^{n_0}+\cdots+b_{-1}t+b_0+\frac{b_1}{t}+\cdots+\frac{b_k}{t^k}+\cdots,$$

set $|A|=2^{-s}$*, where* $s\geq -n_0$ *is the smallest integer such that* $b_s\neq 0$*.*

- By abuse of notation, denote by $\langle A\rangle$ the fractional part of A and set $\mathrm{dist}\,(A,\mathbb{K}[t])=|\langle A\rangle|$.

Conjectures	$\alpha,\beta\in\mathbb{R}$	$A,B\in\mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	$\inf_{q\in\mathbb{Z}\backslash\{0\}}\|q\|\cdot\|\langle q\alpha\rangle\|\cdot\|\langle q\beta\rangle\|=0$	$\inf_{Q\in\mathbb{K}[t]\backslash\{0\}}\|Q\|\cdot\|\langle QA\rangle\|\cdot\|\langle QB\rangle\|=0$
p–adic Littlewood	$\inf_{n\in\mathbb{Z}\backslash\{0\},\,k\geq 0}\|n\|\cdot\left\|\langle p^k n\alpha\rangle\right\|=0$	$\inf_{N\in\mathbb{K}[t]\backslash\{0\},\,k\geq 0}\|N\|\cdot\left\|\langle P(t)^k NA\rangle\right\|=0$

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The state of the art

- Hausdorff dimension of the set **NOT** satisfying the conjecture :

Conjectures	over $\mathbb{R}$	over $\mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	0 (E–Ka–L, 2006)	? ? ? (E–L–M, 2020)
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E : Einsiedler, Ka : Katok, L : Lindenstrauss, Kl : Kleinbock, M : Mohammadi

Figure – Manfred Einsiedler, Anatol Katok, Dmitri Kleinbock, Elon Lindenstrauss & Amir Mohammadi

The state of the art (bis)

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- Validity of the conjectures :

Conjectures	over $\mathbb{R}$	over $\mathbb{K}\left(\left(t^{-1}\right)\right)$
Classical Littlewood	? ? ?	? ? ?
p–adic Littlewood	? ? ?	false when char$(\mathbb{K}) \equiv 3 \pmod 4$

Outline

Statement of the main result

Theorem (A., Badziahin, —)

Let $(p_n)_{n\geq 1} \in \{-1, 1\}^{\mathbb{N}}$ be the paperfolding sequence, let $l \equiv 3 \pmod 4$ be a prime and let $P(t) \in \mathbb{F}_l[t]$ be irreducible. Then, the Laurent series

$$\Pi_P = \sum_{n=1}^{\infty} \frac{p_n}{P(t)^n} \in \mathbb{F}_l\left(\left(t^{-1}\right)\right)$$

is a counterexample to the $P(t)$–adic Littlewood conjecture over $\mathbb{K} = \mathbb{F}_l$:

$$\inf_{\substack{N \in \mathbb{K}[t]\setminus\{0\} \\ k \geq 0}} |N| \cdot \left|\langle P(t)^k N \Pi \rangle\right| = 2^{-4 \cdot \deg P} > 0.$$

Figure – Dzmitry Badziahin (University of Sydney)

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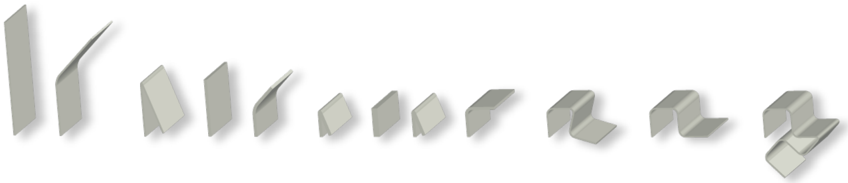
- (Robertson, —) : it is enough to prove the case $P(t) = t$.

The paperfolding sequence



- The paperfolding sequence $(p_n)_{n\geq 0}$ can be obtained by discrete self–similarity :

$$\begin{array}{cccccccccc} (p_0, & p_1, & p_2, & p_3, & p_4, & p_5, & p_6, & p_7, & p_8, & \dots) \\ & & & & = & & & & & \\ (1, & p_0, & -1, & p_1, & 1, & p_2, & -1, & p_3, & 1, & \dots) \end{array}$$

Outline

Reduction of the problem to the existence of singular matrices

- Let

$$A = \sum_{n=1}^{\infty} b_n t^{-n}$$

be a Laurent series and let

$$N = \theta_h t^h + \cdots + \theta_1 t + \theta_0$$

be a polynomial with degree $h \geq 1$ (i.e. $|N| = 2^h$).

- Given $l \geq 1$, asking that $|N| \cdot \left|\langle t^k NA \rangle\right| < 2^{-r}$, i.e.

$$\left|\langle t^k NA \rangle\right| < 2^{-(r+h)},$$

amounts to imposing that a certain number of the coefficients of the Laurent series $t^k NA$ should vanish. Explicitly, one requires that

$$H_A \boldsymbol{\theta} = \mathbf{0},$$

where $H_A = H_A(k, r, h)$ is some square Hankel matrix formed from the coefficients of A, and where $\boldsymbol{\theta} = (\theta_h, \ldots, \theta_1, \theta_0)^T \in \mathbb{K}^{h+1} \backslash \{\mathbf{0}\}$.

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Reduction of the problem to the existence of singular matrices (bis)

- From properties of Hankel matrices, the condition $|N| \cdot \left|\langle t^k NA \rangle\right| < 2^{-r}$ amounts to asking that there should exist r nested singular submatrices inside the infinite Hankel matrix of the sequence $(b_n)_{n \geq 1}$.

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	$\rightarrow$
b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	$\iddots$
b_3	b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	$\iddots$
b_4	b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	$\iddots$
b_5	b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	$\iddots$
b_6	b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	$\iddots$
b_7	b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	$\iddots$
b_8	b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	$\iddots$
b_9	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	$\iddots$
b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	b_{15}	b_{16}	b_{17}	b_{18}	b_{19}	$\iddots$
$\downarrow$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\iddots$	$\searrow$

The Number Wall of a sequence

Definition

The Number Wall of the sequence $\mathcal{B} = (b_n)_{n\geq 1} \in \mathbb{K}^{\mathbb{N}}$ is an infinite matrix $W(\mathcal{B}) = (w_{mn}(\mathcal{B}))_{m\geq 0, n\in\mathbb{Z}}$ whose $(m, n)^{th}$ coefficient is (up to the sign) the Hankel determinant

$$w_{mn}(\mathcal{B}) = (-1)^{m(m-1)/2} \cdot \begin{vmatrix} b_{n-m} & \cdots & b_k & \cdots & b_n \\ \vdots & \iddots & & \iddots & \vdots \\ b_k & & b_n & & b_l \\ \vdots & \iddots & & \iddots & \vdots \\ b_n & \cdots & b_l & \cdots & b_{n+m} \end{vmatrix}.$$

(In particular, when $m = 0$, $w_{0,n}(\mathcal{B}) = b_n$ for all $n \in \mathbb{Z}$.)

The Number Wall of the paperfolding sequence over $\mathbb{F}_3$

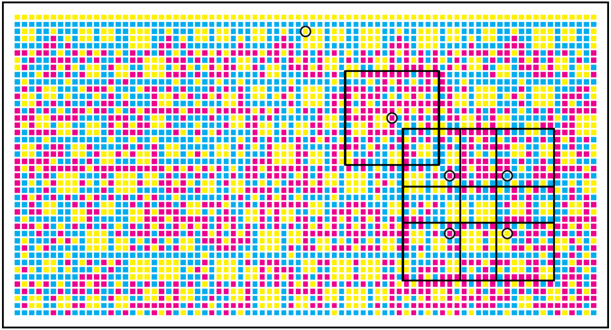

Figure – The Number Wall of the paperfolding sequence over $\mathbb{F}_3$. In yellow (resp. in blue, in pink) the determinants equal to 0 (resp. to 1, to 2).

- From the so–called Desnanot–Jacobi determinental identity, the zero coefficients in this Number Wall can only appear inside squares.

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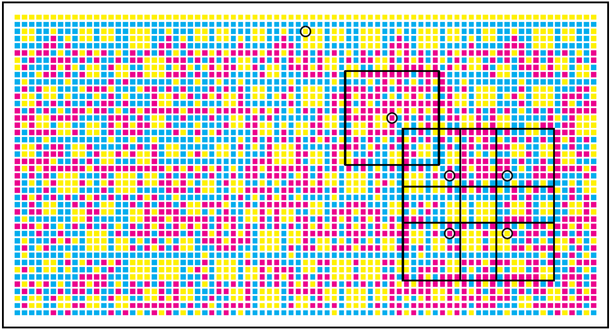

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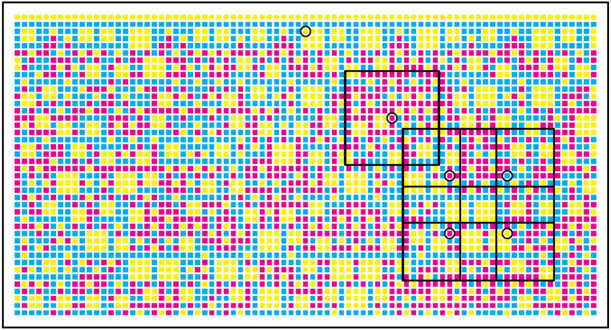

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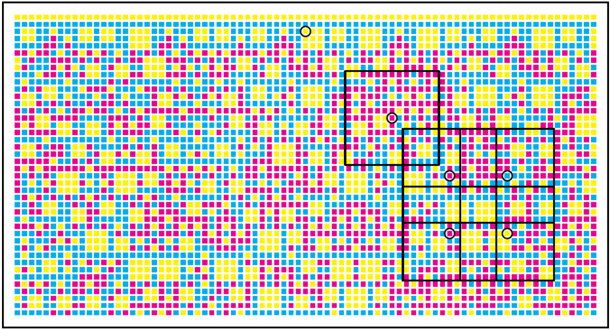
- Reformulation of the problem :

$$\inf_{\substack{N\in\mathbb{K}[t]\setminus\{0\} \\ k\geq 0}} |N|\cdot\left|\langle t^k N\Pi\rangle\right| = 2^{-4} \iff \begin{cases} \text{there exists no } 4\times 4 \text{ zero square} \\ \text{and there exists one such } 3\times 3 \text{ square.} \end{cases}$$

The Number Wall of the paperfolding sequence



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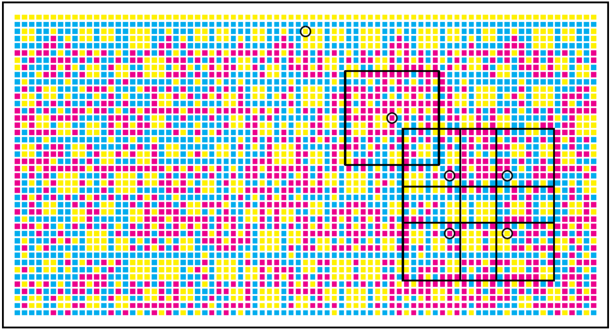
- Computer–assisted proof (A., Nesharim & Lunnon, 2021) :
  1. Tile a finite portion of the Number Wall of the paperfolding sequence ;
  2. Consider the infinite matrix given by this tiling rules ;
  3. Show this is a Number Wall.

The Number Wall of the paperfolding sequence



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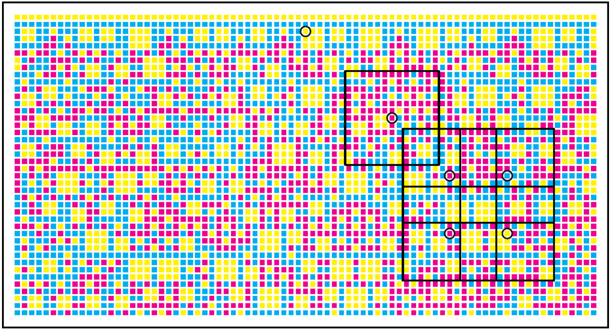

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Implementing the Computer–assisted Proof

- The t–adic (hence the $P(t)$–adic) Littlewood Conjecture fails :
  - over $\mathbb{F}_3$ (A., Nesharim & Lunnon, 2021) ;
  - over $\mathbb{F}_7$ and $\mathbb{F}_{11}$ (Robertson, 2025).

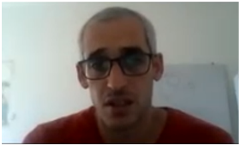


Figure – Erez Nesharim & Fred Lunnon

Conjecture

The Number Wall of a(n one–dimensional) automatic sequence is a (two–dimensional) automatic sequence.

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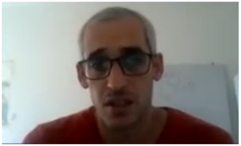


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Outline

Hankel Matrices

- Let $H(m,n)$ be the Hankel determinant

$$H(m,n) = \begin{vmatrix} p_m & \cdots & p_k & \cdots & p_{m+n-1} \\ \vdots & \iddots & & \iddots & \vdots \\ p_k & & p_n & & p_l \\ \vdots & \iddots & & \iddots & \vdots \\ p_{m+n-1} & \cdots & p_l & \cdots & p_{m+2n-1} \end{vmatrix}.$$

- For instance :

$$H(2,5) = \begin{vmatrix} -1 & p_1 & 1 & p_2 & -1 \\ p_1 & 1 & p_2 & -1 & p_3 \\ 1 & p_2 & -1 & p_3 & 1 \\ p_2 & -1 & p_3 & 1 & p_4 \\ -1 & p_3 & 1 & p_4 & -1 \end{vmatrix}.$$

Hankel Matrices

- Let $H(m,n)$ be the Hankel determinant

$$H(m,n) = \begin{vmatrix} p_m & \cdots & p_k & \cdots & p_{m+n-1} \\ \vdots & ⋰ & & ⋰ & \vdots \\ p_k & & p_n & & p_l \\ \vdots & ⋰ & & ⋰ & \vdots \\ p_{m+n-1} & \cdots & p_l & \cdots & p_{m+2n-1} \end{vmatrix}.$$

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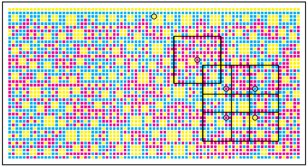
The key recurrence relations

Theorem

There exists sequences $(G(m,n))_{m,n}$ *and* $(F(m,n))_{m,n}$ *such that :*

- $H(2m+1, 2n)$
  $$= (-1)^n \cdot (H(m,n) \cdot H(m+1,n) + G(m,n-1) \cdot G(m+1,n-1))\,;$$
- $H(2m+1, 2n-1)$
  $$= H(m,n) \cdot H(m+1,n-1) - G(m,n-1) \cdot G(m+1,n-2);$$
- $H(2m,2n) \;=\; -\left((H(m,n))^2 + (G(m,n-1))^2\right)\,;$
- $H(2m,2n-1) \;=\; \pm\,(F(m,n-1))^2\,.$



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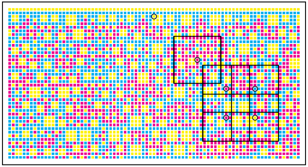

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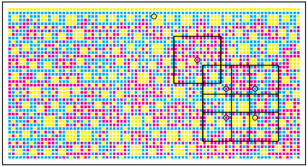
- $H(2m, 2n-1) \; = \; (F(m,n-1))^2\,.$

- **Since -1 is not a quadratic residue modulo $l \equiv 3 \pmod 4$**,

$(H(2m,2n) \equiv 0 \pmod l) \implies (H(m,n) \equiv 0 \pmod l) \;\&\; G(m,n-1) \equiv 0 \pmod l)$

(otherwise $\left(G(m,n-1) \cdot H(m,n)^{-1}\right)^2 \equiv -1 \pmod l$ — contradiction.)

Key Steps in the Completion of the Proof



Step 1 : The recurrence relations imply that :

- A zero square with length $r \geq 2$, say $W(2m, 2n)$, arises from one with smaller index, say $W(m, n)$;
- If the smaller one $W(m, n)$ has sidelength $r \geq 2$, then the larger one $W(2m, 2n)$ has length $2r - 3$.

Step 2 : To prove that $r \leq 3$, use the theory of continued fractions :

- it boils down to to showing that the denominators of the convergents of (a family derived from) the paperfolding Laurent series $\Pi = \sum_{n=1}^{\infty} p_n \cdot t^{-n}$ are never divisible by $t^2 + 1$.
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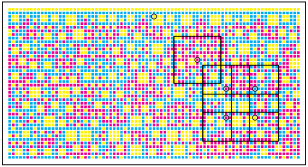

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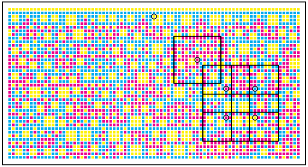

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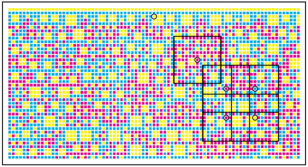

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The $P(t)$–adic Littlewood Conjecture in other characteristics

- Recall the definition of the paperfolding sequence :

$$\begin{array}{cccccccccc} (p_0, & p_1, & p_2, & p_3, & p_4, & p_5, & p_6, & p_7, & p_8, & \ldots) \\ & & & & = & & & & & \\ (1, & p_0, & -1, & p_1, & 1, & p_2, & -1, & p_3, & 1, & \ldots) \end{array}$$

Conjecture (Robertson, 2025)

Let l be an odd prime and let s be the largest power of 2 dividing $l-1$. Then the sequence obtained from the above by applying the block substitutions

$$\begin{aligned} 1 &\mapsto 1, 2, \ldots, s \\ -1 &\mapsto -1, -2, \ldots, -s \end{aligned}$$

fails the $P(t)$–adic Littlewood Conjecture over any field $\mathbb{K}$ s.t. $char(\mathbb{K}) = l$.

- when $s = 1$, $l \equiv 3 \pmod 4$ and the sequence is the paperfolding sequence.

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- Robertson (2025) : the conjecture is true when $l = 5$ (through the computer–assisted method).

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- No clear understanding of the case $\mathrm{char}(\mathbb{K}) = 2$.

The end