TALK 9

$X/\mathbb{C}$ smooth algebraic variety

$Z \subseteq X$: codimension p algebraic subset

$$\Rightarrow GE^p_Z(X) := \{\tilde{g} \in \tilde{E}^{p-1,p-1}_{\log,\mathbb{R}}(X, Z) : \partial\bar{\partial} g \in E^{p,p}_{\log,\mathbb{R}}(X)\}$$

$$\alpha \in Z^p(X) \Rightarrow GE^p_\alpha(X) := \{\tilde{g} \in GE^p_{|\alpha|}(X) : cl(\tilde{g}) = \rho(\alpha)\}$$

Principal divisors $W \xrightarrow{i} X$, codimension $p-1$, $f \in k(W)^\times$.

$$\Rightarrow \frac{1}{2}\log(f\bar{f}) \in \mathcal{D}'(E^*_{\log}(W, |div(f)|), 1)$$

$$\Rightarrow \rho(f) := i_!\left(\frac{1}{2}\log(f\bar{f})\right) \in H^{2p-1}_{\mathcal{D}}(X, |div(f)|, \mathbb{R}(p))$$

$$\Rightarrow b(\rho(f)) \in GE^p_{div(f)}(X), \ \widehat{div}(f) = (div(f), -b(\rho(f)))$$

$$\Rightarrow \widehat{CH}^p(X) = \frac{\{(z,g) \mid z \in Z^p(X), g \in GE^p_z(X_{\mathbb{R}})\}}{\{\widehat{div}(f) \mid f \in k(W)^\times, W \subseteq X \text{ codim. } p-1\}}$$

defined over arithmetic ring R.

Note that $f \in k(W)^\times$ defines a cycle $\Gamma_f \subseteq W \times \mathbb{P}^1$ of codimension one (the graph) and therefore a cycle in $X \times \mathbb{A}^1$ of codimension p. These cycles all together define $CH^{p,p-1}(X)$, with a map $\rho: CH^{p,p-1}(X) \to \tilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}})$, the Beilinson regulator. We also have the Beilinson regulator in degree (p,p), giving $\rho: CH^p(X) \to H^{2p}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p))$. Furthermore, we have the following maps:

$$\widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X), \quad \tilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X),$$
$$\{(z,\tilde{g})\} \mapsto \{z\} \qquad \tilde{g} \longmapsto (0,\tilde{g})$$

$$\widehat{CH}^p(X) \xrightarrow{\omega} ZE^{p,p}_{\log}(X_{\mathbb{R}}), \quad ZE^{p,p}_{\log}(X_{\mathbb{R}}) \xrightarrow{h} H^{2p}_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(p))$$
$$\{(z,\tilde{g})\} \mapsto -2\partial\bar{\partial} g \qquad \alpha \longmapsto \{\alpha\}$$

and we define $CH^p(X)_0 := \ker(\rho)$, $\widehat{CH}^p(X)_0 := \ker(\omega)$.
Then, we have the following exact sequences:

$$CH^{p,p-1}(X) \xrightarrow{\rho} \widetilde{E}^{p-1,p-1}_{\log}(X_{\mathbb{R}}) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$$

$$CH^{p,p-1}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{(\zeta,-\omega)} CH^p(X) \oplus ZE^{p,p}_{\log}(X_{\mathbb{R}}) \xrightarrow{\rho+h} H^{2p}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) \to 0$$

$$CH^{p,p-1}(X) \xrightarrow{\rho} H^{2p-1}_{\mathcal{D}}(X_{\mathbb{R}},\mathbb{R}(p)) \xrightarrow{a} \widehat{CH}^p(X)_0 \xrightarrow{\zeta} CH^p(X)_0 \to 0$$

which imply that $\rho \circ \zeta = h \circ \omega$.

$\boxed{\text{Theorem}}$ If $E \xrightarrow{\pi} X$ is a vector bundle, and $\widehat{CH}^p(E)_{vert} := \omega^{-1}(\pi^* ZE^{p,p}_{\log}(X_{\mathbb{R}}))$ then $\pi^*: \widehat{CH}^p(X) \to \widehat{CH}^p(E)_{vert}$ is an isomorphism.

$\boxed{\text{Proof}}$ Follows from the exact sequences, the five lemma, and the fact that $CH^p(-)$, $CH^{p,p-1}(-)$, $H^i_{\mathcal{D}}(-,\mathbb{R}(j))$ are all homotopy invariant. ∎

DEFORMATIONS AND GREEN FORMS

$X/\mathbb{C}$: smooth, quasiprojective, $\dim(X) = n$,

$T: X \to \mathbb{P}^1$: "degeneration", with:

- $T^{-1}(t)$ smooth, $\forall t \in \mathbb{P}^1(\mathbb{C}) \setminus \{0\}$
- $T^{-1}(0) = D_1 \cup D_2$ for $D_1, D_2 \subseteq X$ divisors with SNC

Fix $D_3 \subseteq X$ divisor with SNC such that $D_1 \cup D_2 \cup D_3$ is SNCD and D_1, D_2, D_3 have no common components

Notation $D \subseteq X$: SNCD $\to E_{cX}^{p,q}(\text{null } D) := \{\omega \in E_{cX}^{p,q} \mid \omega|_D = 0\}$

$\mathcal{D}^{p+1,q+1}_{X/D}$: topological dual of $E^{p,q}_{cX}(\text{null } D)$.

In the previous setting, fix $\omega \in E_X^{p,q}(\log D_1 \cup D_3)$ and consider the map:

$$\mathbb{P}^1(\mathbb{C}) \setminus \{0\} \to \mathcal{D}^{p+1,q+1}_{X/D_3},\quad t \mapsto \omega \wedge \delta_{T^{-1}(t)}$$

or, in other words, if $\varphi \in E_X^{n-p-1,n-q-1}(\text{null } D_3)$, look at

$$\mathbb{P}^1(\mathbb{C}) \setminus \{0\} \to \mathbb{C},\quad t \mapsto \int_{T^{-1}(t)} \omega \wedge \varphi$$

and study the asymptotics of this function as $t \to 0$.

Theorem There exists $N \in \mathbb{N}$ and continuous functions $C_k : \mathbb{C} \to \mathcal{D}^{p+1,q+1}_{X/D_3}$ such that:

$$\omega \wedge \delta_{T^{-1}(t)} = \sum_{k=0}^{N} (\log|t|)^k \, C_k(t) \text{ for all } t \in \mathbb{C}^\times.$$

Moreover, $\forall \varphi \in E_X^{n-p-1,n-q-1}(\text{null } D_3)$ we have that:

- $t \mapsto C_k(t)(\varphi)$ is smooth on $\mathbb{C}^\times$
- $t \mapsto C_k(t)(\varphi)$ is α-Hölder [redacted] continuous at 0, i.e. $|C_k(t)(\varphi)| \le K \cdot |t|^\alpha$ for all $t \in \mathbb{C}^\times$ small enough.

Furthermore, $C_k(0)$ is independent of T, and $C_k(0)(\varphi)$ depends only on $\varphi|_{D_1 \cup D_2}$. Finally, if ω is locally integrable on X, then $C_k(t) \in \mathcal{D}_X^{p+1,q+1}$.

Proof We first deal with a local case. Since $D_1 \cup D_2 \cup D_3$ is a SNCD, we have locally that $X = \Delta^n$ with $D_1 := \{z_1 \cdots z_m = 0\}$, $D_2 := \{z_{m+1} \cdots z_{m+r} = 0\}$, and $D_3 := \{z_{m+r+1} \cdots z_{m+r+s} = 0\}$. Furthermore, $T(z_1, \dots, z_n) = z_1 \cdots z_{m+r}$.

Now, we can assume WLOG that $D_3 = \emptyset$. Indeed, if $\varphi \in E_X^{n-p-1,n-q-1}(\mathrm{null}\, D_3)$ then $\varphi = \|z_{m+r+1} \cdots z_{m+r+s}\|^2 \varphi'$ for some $\varphi' \in E_X^{n-p-1,n-q-1}$. Furthermore, since $\omega \in E_X^{p+1,q+1}(\log D_1 \cup D_3)$ we can write

$$\omega = \sum_{\substack{I,J,k \\ \{1,\dots,s\}}} \omega_{I,J,k} \cdot \left(\prod_{k \in K} (\log|z_{m+r+k}|)^{a_k}\right)\left(\prod_{\substack{i \in I \\ j \in J}} \frac{dz_{m+r+i}}{z_{m+r+i}} \wedge \frac{d\bar{z}_{m+r+j}}{\bar{z}_{m+r+j}}\right)$$

where $\omega_{I,J,k} \in E_X^*(\log D_1)$. This implies that

$$\int_{T^{-1}(t)} \omega \wedge \varphi = \int_{z_1 \cdots z_{m+r} = t} \omega \wedge \varphi = \int_{\substack{(z_1, \dots, z_{m+r}) \\ z_1 \cdots z_{m+r} = t}} \left(\int_{\Delta^{n-m-r}} \omega \wedge \varphi \right),$$ where the

integral inside is a sum of integrals of the form

$$\int_{(z_{m+r+1}, \dots, z_n)} f \cdot \left(\prod (\log|z_k|)^{a_k}\right) d\mathrm{vol}_{\Delta^{n-m-r}}$$ where

f is smooth. Since $(\log|x|)^a$ is integrable, the above integral is smooth as a function of $(z_1, \dots, z_{m+r})$, and therefore will not affect the conclusions of the theorem. Therefore we can assume that $D_3 = \emptyset$, as we claimed.

Now, take $D_1 : \{z_1 \cdots z_m = 0\}$, $D_2 : \{z_{m+1} \cdots z_{m+r} = 0\}$, $T(z_1, \dots, z_n) = z_1 \cdots z_{m+r}$, $\omega \in E_X^{p+1,q+1}(\log D_1)$ and

$\varphi \in \mathcal{E}_{c\,X}^{n-p-1,\,n-q-1}$. Then, we can write $z_j = r_j e^{i\theta_j}$ and we can perform the integral $\int\limits_{z_1\cdots z_{m+r}=t} \omega \wedge \varphi$ first over the θ_j's and then over the r_j's. This shows that

$$\int\limits_{z_1\cdots z_{m+r}=t} \omega\wedge\varphi = \sum_{j=1}^{n} \int\limits_{\substack{(r_1,\dots,r_n)\in[0,1]^n \\ r_1\cdots r_{m+r}=|t|}} g_j(\log r_1,\dots,\log r_{m+r})\,\frac{dr_1}{r_1}\cdots\widehat{\frac{dr_j}{r_j}}\cdots\frac{dr_n}{r_n}$$

where $g_j \in \mathcal{C}^\infty_c([0,1]^n)[y_1,\dots,y_m]$.

Now, suppose that $a \in C_c^\infty([0,1)^n)$, and fix $l_1,\dots,l_m \in \mathbb{N}$. Then, the function

$$F(\tau) := \int\limits_{\substack{(r_1,\dots,r_n)\in[0,1)^n \\ r_1\cdots r_{m+r}=\tau}} a(r_1,\dots,r_n)\prod_{j=1}^{m}\log(r_j)^{l_j}\,dr_1\cdots dr_{n-1}$$

if $m<n$

is α-Hölder continuous at $\tau=0$. Indeed:

- if $m+r<n$ then $F(0)=0$, while

$$|F(\tau)| \le \|a\|_\infty \cdot \int\limits_{r_1\cdots r_{m+r}=\tau} \prod_{j=1}^{m}|\log(r_j)|^{l_j}\,dr_1\cdots dr_{m+r}$$

$$\le \|a\|_\infty \cdot \sum_{j=1}^{m+r} \int\limits_{\substack{(r_1,\dots,r_n) \\ r_j\le \tau^{1/(m+r)}}} \prod \log(r_j)^{l_j}\,dr_1\cdots dr_{m+r}$$

$$\le C_1\,\tau^{\frac{1}{(m+r)}}\cdot P(\log\tau),\quad P\in\mathbb{R}[T].$$

6| Therefore $|F(\bar{\tau})| \leq C_1' \cdot \bar{\tau}^{\alpha}$, $\forall \alpha \in \left(0, \frac{1}{m+v}\right)$.

• if $m+v=n$, then $F(0) = \displaystyle\int_{(r_1,\dots,r_{n-1})} a(r_1,\dots,r_{n-1},0)\left(\prod_{j=1}^{m} \log(r_j)^{l_j}\right) dr_1 \dots dr_{n-1}$

[redacted] Moreover, if $\bar{\tau} \neq 0$ then

$$F(\bar{\tau}) = \int_{(r_1,\dots,r_{n-1})} a\left(r_1,\dots,r_{n-1},\frac{\bar{\tau}}{r_1 \cdots r_{n-1}}\right)\left(\prod_{j=1}^{m} \log(r_j)^{l_j}\right) dr_1 \dots dr_{n-1}$$

and therefore we have that

$$F(\bar{\tau}) - F(0) = \int_{(r_1,\dots,r_{n-1})} \left(a\left(r_1,\dots,r_{n-1},\frac{\bar{\tau}}{r_1 \cdots r_{n-1}}\right) - a(r_1,\dots,r_{n-1},0)\right)\left(\prod_{j=1}^{m} \log(r_j)^{l_j}\right) dr_1 \dots dr_{n-1}$$

Now, we can split the integral into the regions $\prod_{j=1}^{n-1} r_j \leq \bar{\tau}^{1/2}$ and $\prod_{j=1}^{n-1} r_j \geq \bar{\tau}^{1/2}$. On the first region, we have as before that

$$\left| \int_{\prod_{j=1}^{n-1} r_j \leq \bar{\tau}^{1/2}} \left(a\left(r_1,\dots,r_{n-1},\frac{\bar{\tau}}{r_1 - r_{n-1}}\right) - a(r_1,\dots,r_{n-1},0)\right) \prod_{j=1}^{m} \log(r_j)^{l_j} dr_1 \dots dr_{n-1} \right|$$

$$\leq 2 \cdot \|a\|_\infty \cdot \int_{r_1 \cdots r_{n-1} \leq \bar{\tau}^{1/2}} \prod_{j=1}^{m} |\log(r_j)|^{l_j} dr_1 \dots dr_{n-1}$$

$$\leq 2 \cdot \|a\|_\infty \sum_{j=1}^{n-1} \int_{r_j \leq \bar{\tau}^{1/(2(n-1))}} \prod_{j=1}^{m} |\log(r_j)|^{l_j} dr_1 \dots dr_{n-1}$$

$\le C_2 \cdot \tau^{1/2(n-1)} \cdot P_2(\log(\tau))$. On the other hand

$$\left| \int_{r_1 \cdots r_{n-1} \ge \tau^{1/2}} \left(a\left(r_1, \ldots, r_{n-1}, \frac{\tau}{r_1 \cdots r_{n-1}}\right) - a(r_1, \ldots, r_{n-1}, 0) \right) \prod_{j=1}^{m} \log(r_j)^{l_j} \, dr_1 \ldots dr_{n-1} \right|$$

$$\le \left\| \frac{\partial a}{\partial r_n} \right\|_\infty \cdot \int_{r_1 \cdots r_{n-1} \ge \tau^{1/2}} \frac{\tau}{r_1 \cdots r_{n-1}} \left(\prod_{j=1}^{m} |\log(r_j)|^{l_j} \right) dr_1 \ldots dr_{n-1}$$

$$\le \left\| \frac{\partial a}{\partial r_n} \right\|_\infty \cdot \tau^{1/2} \cdot \prod_{j=1}^{m} (l_j)!$$ Therefore

$$|F(\tau) - F(0)| \le C_2 \, \tau^{1/2(n-1)} P_2(\log(\tau)) + C_2' \, \tau^{1/2}$$
$$\le C_2'' \cdot \tau^\alpha, \quad \forall \alpha \in \left(0, \frac{1}{2(n-1)}\right),$$

if τ is small enough. Now, let us consider instead the function

$$F(\tau) = \int_{\substack{(r_1, \ldots, r_n) \in [0,1)^n \\ r_1 \cdots r_n = \tau}} a(r_1 \ldots r_n) \left(\prod_{j=1}^{n} \log(r_j)^{l_j} \right) \frac{dr_1}{r_1} \cdots \frac{dr_{n-1}}{r_{n-1}}$$

and let us prove that, when $\tau \to 0$, this function admits the asymptotic expansion

$$F(\tau) = \sum_{k=0}^{N} f_k(\tau) \log(\tau)^k$$

where $f_k : [0,1) \to \mathbb{R}$ is smooth on $(0,1)$ and -Hölder at 0. To do so, we consider the functions

$$F_i(\tau) = \int_{\substack{(r_1,\dots,r_n)\in[0,1)^n \\ r_1\cdots r_n=\tau}} a(r_1\dots r_n)\left(\prod_{j=1}^{n}\log(r_j)^{l_j}\right)\frac{dr_1}{r_1}\cdots\frac{dr_i}{r_i}\,dr_{i+1}\cdots dr_{n-1}$$

and we proceed by induction on i.

$\boxed{i=0}$ $$F_0(\tau) = \int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_n)\left(\prod_{j=1}^{n}\log(r_j)^{l_j}\right)dr_1\cdots dr_{n-1}$$

Since $r_1\cdots r_n=\tau$ then $\log(r_1)+\dots+\log(r_n)=\log(\tau)$. Replacing $\log(r_n)$ with $\log(\tau)-\log(r_1)-\dots-\log(r_{n-1})$ we see that

$$F_0(\tau) = \int_{(r_1,\dots,r_{n-1})\in[0,1)^{n-1}} a\left(r_1,\dots,r_{n-1},\frac{\tau}{r_1\cdots r_{n-1}}\right)\left(\prod_{j=1}^{n-1}\log(r_j)^{l_j}\right)\left(\sum (-1)^{k_1+\dots+k_{n-1}}\log(\tau)^{k_0}\cdots\log(r_{n-1})^{k_{n-1}}\binom{l_n}{k_0\cdots k_{n-1}}\right)dr_1\cdots dr_{n-1}$$

$$=\sum_{\substack{k_0,\dots,k_{n-1}\\ k_0+\dots+k_{n-1}=l_n}}(-1)^{k_1+\dots+k_n}\binom{l_n}{k_0,\dots,k_{n-1}}\left(\int_{(r_1,\dots,r_{n-1})\in[0,1)^{n-1}} a\left(r_1,\dots,r_{n-1},\frac{\tau}{r_1\cdots r_{n-1}}\right)\left(\prod_{j=1}^{n-1}\log(r_j)^{l_j+k_j}\right)dr_1\cdots dr_{n-1}\right)\log(\tau)^{k_0}$$

and each one of the integrals appearing in the sum is α-Lipschitz by the previous result.

$\boxed{i-1\Rightarrow i}$ $$F_i(\tau) = \int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_n)\left(\prod_{j=1}^{n}\log(r_j)^{l_j}\right)\frac{dr_1\cdots dr_i}{r_1\cdots r_i}\,dr_{i+1}\cdots dr_{n-1}$$

$$=\int_{r_1\cdots r_n=\tau}\left(a(r_1,\dots,r_n)-a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_n)\right)\left(\prod_{j=1}^{n}\log(r_j)^{l_j}\right)\frac{dr_1\cdots dr_i}{r_1\cdots r_i}\,dr_{i+1}\cdots dr_{n-1}$$

$$+\underbrace{\int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_n)\left(\prod_{j=1}^{n}\log(r_j)^{l_j}\right)\frac{dr_1\cdots dr_i}{r_1\cdots r_i}\,dr_{i+1}\cdots dr_{n-1}}_{(II)}$$

The first integral can be rewritten as

$$\int_{r_1\cdots r_n=\tau}\underbrace{\left(\frac{a(r_1,\dots,r_n)-a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_n)}{r_i}\right)}_{=:\tilde{a}(r_1,\dots,r_n)}\left(\prod\log(r_j)^{l_j}\right)\frac{dr_1\cdots dr_{i-1}}{r_1\cdots r_{i-1}}\,dr_i\cdots dr_{n-1}$$

where $\tilde{a}$ is smooth on $[0,1)^n$. Therefore ~~we~~ this integral has the required asymptotic expansion by induction. [illegible]

$$\cancel{\int_{r_1\cdots r_n=\tau}\overbrace{\tilde{a}(r_1,\dots,r_i,0,r_{i+1},\dots,r_n)}^{=\tilde{a}(r_1,\dots,r_i,\dots,r_n)}\left(\prod_{j=1}^{n}\log(r_j)^{l_j}\right)\frac{dr_1\cdots dr_i}{r_1\cdots r_i}\,dr_{i+1}\cdots dr_{n-1}}$$

$$\cancel{=\int_\tau^1 \log(r_i)^{l_i}\underbrace{\left(\int_{r_1\cdots\hat{r}_i\cdots r_n=\tau/r_i}\tilde{a}\cdot\prod^{\wedge}\log(r_j)^{l_j}\,\frac{dr_1\cdots dr_{i}\cdots dr_{n-1}}{r_1\cdots\hat{r}_i\cdots r_{i}}\right)}_{=:g(\tau/r_i)}\frac{dr_i}{r_i}}$$

By induction, $g(\tau/r_i)=$

~~For the second integral, note that if $(r_1,\dots,r_n)\in[0,1)^n$ and $r_1\cdots r_n=\tau$ then:~~

~~$\tau\le r_1\le 1$, $\tau/r_1\le r_2\le 1$, $\dots$, $\tau/(r_1\cdots\hat{r}_i\cdots r_{n-1})\le r_i\le 1$,~~

~~and then $r_n=\tau/(r_1\cdots r_{n-1})$. Therefore we can write~~

$$\cancel{\int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_n)\prod\log(r_j)^{l_j}\frac{dr_1\cdots dr_i\cdots dr_{n-1}}{r_1\cdots r_i}} =$$

$$\cancel{=\int_{\tau}\int_{\tau/r_1}\cdots\left(\int_{\tau/r_1\cdots \hat{r}_i\cdots r_{n-1}}\log(r_i)^{l_i}\frac{dr_i}{r_i}\right)\frac{dr_1\cdots \hat{dr_i}\cdots dr_{n-1}}{r_1\cdots r_{i-1}}} =$$

$$=$$

For the second integral, note first of all that $d(r_1\cdots r_n)=0$, hence $\frac{dr_i}{r_i} = -\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{dr_j}{r_j}$. Therefore

$$II = -\int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_n)\prod_{j=1}^{n}\log(r_j)^{l_j}\frac{dr_1}{r_1}\cdots\frac{dr_{i-1}}{r_{i-1}}dr_{i+1}\cdots dr_{n-1}\frac{dr_n}{r_n}$$

Adding and substracting $a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_{n-1},0)$, we are left with the integral:

$$\int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_{n-1},0)\prod_{j=1}^{n}\log(r_j)^{l_j}\frac{dr_1\cdots dr_{i-1}}{r_1\cdots r_{i-1}}dr_{i+1}\cdots dr_{n-1}\frac{dr_n}{r_n}$$

$$=-\int_{r_1\cdots r_n=\tau} a(r_1,\dots,r_{i-1},0,r_{i+1},\dots,r_{n-1},0)\prod_{j=1}^{n}\log(r_j)^{l_j}\frac{dr_1\cdots dr_i}{r_1\cdots r_i}dr_{i+1}\cdots dr_{n-1}$$

Substituting $r_n = \tau/r_1\cdots r_{n-1}$ we are left with:

$$\int_{\tau\le r_1\cdots r_{n-1}\le 1}\tilde{a}(r_1,\dots,r_i,0,\dots,r_{n-1})\prod_{j=1}^{n-1}\log(r_j)^{k_j}\frac{dr_1\cdots dr_i}{r_1\cdots r_i}dr_{i+1}\cdots dr_{n-1}$$

$$= \int_{\tau \le r_1 \cdots \hat{r}_i \cdots r_{n-1} \le 1} \tilde{a}(r_1, \ldots, r_{i-1}, 0, \ldots, r_{n-1}) \prod_{\substack{j=1 \\ j \ne i}}^{n-1} \log(r_j)^{k_j} \left(\int_{\tau / r_1 \cdots \hat{r}_i \cdots r_{n-1}} \log(r_i)^{k_i} \frac{dr_i}{r_i} \right) \frac{dr_1 \cdots dr_{i-1}}{r_1 \cdots r_{i-1}} \, dr_{i+1} \cdots dr_{n-1}$$

$$= \int_{\tau \le r_1 \cdots \hat{r}_i \cdots r_{n-1} \le 1} \tilde{a} \cdot \prod \log(r_j)^{k_j} \cdot \left(-\frac{1}{k_j+1} \left(\log \tau - \sum_{\substack{j=1 \\ j \ne i}}^{n-1} \log r_j \right)^{k_j+1} \right) \frac{dr_1 \cdots dr_{i-1}}{r_1 \cdots r_{i-1}} \, dr_{i+1} \cdots dr_n$$

which has a similar asymptotic expansion as before by induction.

Analogously we have similar asymptotics for

$$\int_{r_1 \cdots r_n = \tau} a(r_1, \ldots, r_n) \prod_{j=1}^{n} \log(r_j)^{l_j} \frac{dr_1}{r_1} \cdots \frac{dr_{i-1}}{r_{i-1}} \frac{dr_{i+1}}{r_{i+1}} \cdots \frac{dr_n}{r_n}$$

and therefore also for

$$\int_{z_1 \cdots z_n = t} \omega \wedge \varphi = \sum_{i=1}^{n} \int_{r_1 \cdots r_n = |t|} g_i(\log r_1, \ldots, \log r_n) \frac{dr_1}{r_1} \cdots \widehat{\frac{dr_i}{r_i}} \cdots \frac{dr_n}{r_n}$$

where $g_i \in C_c^\infty([0,1)^n)[T_1, \ldots, T_n]$. Moreover, in the asymptotic expansion for $\int g_i \frac{dr_1 \cdots \widehat{dr_i} \cdots dr_n}{r_1 \cdots \hat{r}_i \cdots r_n}$ the constant term only depends on the values of the coefficients of g_i restricted to $r_i = 0$. Hence

 $G_k(0)(\ell)$ depends only on w and $\ell|_{D_1\cup D_2}$ in this particular way of taking asymptotic expansions. Note moreover that $G_k(0)$ depends uniquely on w, as follows from the following result.

Lemma $\{f_k : [0,1) \to \mathbb{R}\}_{k=0}^{N}$ smooth on $(0,1)$, α-Hölder at $t=0$, and $\lim_{t\to 0}\sum_{k=0}^{N} f_i(t)(\log t)^i = 0 \Rightarrow f_0(0) = \dots = f_N(0) = 0$.

Proof By induction on N. If $N=0$, this is obvious. Otherwise, $f_N(0) = -\lim_{t\to 0} \frac{f_{N-1}(t)}{\log(t)} + \dots + \frac{f_0(t)}{\log(t)^N} = 0$.

Moreover, since f_N is α-Hölder continuous, $\lim_{t\to 0}|f_N(t)(\log t)^N| \leq C \lim_{t\to 0} t^\alpha (\log t)^N = 0$, have

$$\lim_{t\to 0}\sum_{k=0}^{N} f_i(t)(\log t)^i = \lim_{t\to 0}\sum_{k=0}^{N-1} f_i(t)(\log t)^i = 0$$

and by induction $f_0(0) = \dots = f_{N-1}(0) = 0$.

This proves the theorem in the local case. For the global case, we can choose an open cover of X such that on each open U of this cover we have that $U \cong \Delta^n$ and $U \cap (D_1 \cup D_2 \cup D_3) : \{z_1 \cdots z_{a+b+c} = 0\}$. We can then choose a partition of unity $\{\lambda_i\}$ subordinated to $\{U_i\}$ and we get that:

$$w \wedge \delta_{T^{-1}(t)} = \left(\sum_i \lambda_i w\right) \wedge \delta_{T^{-1}(t)} = \sum_i \lambda_i w \wedge \delta_{T^{-1}(t)} =$$

$$= \sum_i \left(\sum_k G_{k,i}(t) \log|t|^k\right) = \sum_k \left(\sum_i G_{k,i}(t)\right)(\log|t|)^k$$

Here, a priori, $G_w(t)$ can be nontrivial $\forall w \geq 1$, because the value of N can be unbounded. However, as we will see, in our case we can assume X projective, and then we can take a finite open cover.

SPECIALIZATION TO THE NORMAL CONE

We are almost ready to prove the core technical statement of Hu's thesis.

$X/\mathbb{C}$: smooth algebraic, $\dim(X) = n$
$Y \subseteq X$: smooth, $Z \subseteq X$: codimension p
$\tilde{g} \in GE^p_Z(X)$. Then, there exists a good compactification $(\overline{X}, D)$ of $X \setminus Z$ such that $\overset{D_1 \cup \dots \cup D_r}{D} \subseteq \overline{X}$ is a SNCD and $g = \sum \alpha_i \lambda_i + \beta$ for some $\alpha_1, \dots, \alpha_r, \beta$ which are smooth on $\overline{X}$, and such that $\alpha_i|_{D_i}$ is ∂ and $\bar{\partial}$-closed, where $\lambda_1, \dots, \lambda_r$ are Green functions for $D_1, \dots, D_r$.
By Hironaka, there exists $\overline{X} \supseteq X$ smooth compactification with $D := \overline{X} \setminus X$ divisor with SNC and $\overline{Y} \subseteq \overline{X}$ smooth, where $\overline{Y}$ is the closure of Y in $\overline{X}$. Now, we can consider $W := Bl_{Y \times \{0\}}(X \times \mathbb{P}^1)$ and $\overline{W} := Bl_{\overline{Y} \times \{0\}}(\overline{X} \times \mathbb{P}^1)$, as well as $\tilde{X} := Bl_Y(X)$ and $\tilde{\overline{X}} := Bl_{\overline{Y}}(\overline{X})$. Then

$$\begin{tikzcd} & \overline{W} \arrow[dl, "\overline{\Gamma}"'] \arrow[dr, "\overline{p}"] & \\ \mathbb{P}^1 & & \overline{X} \\ \mathbb{P}^1 \arrow[u, equal] & W \arrow[uu, hook] \arrow[l, "\Gamma"'] \arrow[r, "p"] & X \arrow[u, hook] \end{tikzcd}$$

$$\Gamma^{-1}(t) = \begin{cases} X, & \text{if } t \neq 0 \\ \mathbb{P}(N_Y(X) \oplus 1) \cup \tilde{X}, & \text{if } t = 0 \end{cases}$$

while $\overline{T}^{-1}(t) = \begin{cases} \tilde{X}, & \text{if } t \neq 0 \\ \mathbb{P}(N_{\tilde{Y}}(\tilde{X}) \oplus 1) \cup \tilde{X}, & \text{if } t = 0 \end{cases}$

Now, we let $W^\circ := W \setminus \tilde{X}$ be the deformation to the normal cone, and we fix a representative g_Z of a Green form $\tilde{g}_Z \in GE^p_Z(X)$. Then, g_Z has logarithmic singularities along Z and along infinity.

Now, let $\overline{Z} \subseteq \overline{X}$ be the Zariski closure of Z, and let $\overline{X}' \xrightarrow{\pi_X} \overline{X}$ be a resolution of $\overline{Z}$, i.e. a birational map with $\overline{X}'$ smooth and projective such that $\overline{Z}' := \pi_X^{-1}(Z)$ is a divisor with SNC. We can then also resolve the singularities for $\overline{p}^{-1}(Z)$, obtaining $\overline{W}' \xrightarrow{\overline{p}'} \overline{X}'$ such that $(\overline{p}')^{-1}(\overline{Z}' \cup \pi_X^{-1}(D))$, $(\overline{T} \circ \pi_W)^{-1}(0)$ and their union are SNCDs.

Therefore, we can take $D_1 = (\overline{T} \circ \pi_W)^{-1}(0)$, $D_2 = \phi$ and $D_3 = (\overline{p}')^{-1}(\overline{Z}' \cup \pi_X^{-1}(D))$ and we can apply the previous deformation result to the ~~current~~ form $(\overline{p}')^*(g_Z) \in E^{p-1,p-1}_{\overline{W}'}(\log(D_1 \cup D_2))$. We get

$$(\overline{p}')^*(g_Z) \wedge \delta_{(\overline{T}')^{-1}(t)} = \sum G'_k(t)\,(\log|t|)^k$$

and we can then push forward these currents to obtain $G_k(t) \in \mathcal{D}^{p,p}_{\overline{W}/\overline{p}^{-1}(\overline{Z} \cup D)}$, and in fact to get currents $C_k(t) \in \mathcal{D}^{p,p}_{W^0/\overline{Z \times (\mathbb{P}^1(\mathbb{C}) \setminus \{0\})}}$ such that

$$(p^0)^* g_Z \wedge \delta_{(T^0)^{-1}(t)} = \sum C_k(t)\,(\log|t|)^k$$

and for each $\varphi \in {}_c E^{n-p-1, n-p-1}_{\overline{W^0}}(\text{null}(\overline{Z \times \mathbb{P}^1(\mathbb{C}) \setminus \{0\}} \cup \partial W^0))$

the functions $t \mapsto C_k(t)(\varphi)$ are smooth away from zero and α-Hölder at $t=0$. Moreover:

- $C_k(0)$ is uniquely determined by g_Z,
- $C_k(0)(\varphi)$ depends only on $\varphi|_{N_Y(X)}$, which means that $C_k(0) \in \mathcal{D}^{p,p}_{N_Y X / C_{Y \cap Z} Z}$, and moreover $C_k(0) \in GC^p_{|C_{Y \cap Z}(Z)|}(N_Y X)$. Indeed, this follows from the following compatibility between specialization and differentials.

$\boxed{\text{Lemma}}$ $X/\mathbb{C}$ smooth, $D_1, D_2, D_3 \subseteq X$: SNCDs, $\omega \in E^{p,q}_X(\log D_1 \cup D_3)$, $T : X \to \mathbb{P}^1(\mathbb{C})$, $T^{-1}(0) = D_1 \cup D_2$, then if

$$\begin{cases} (\partial\omega) \wedge \delta_{T^{-1}(t)} = \sum_{k=1}^{N} C'_k(t)\,(\log|t|)^k \\ \omega \wedge \delta_{T^{-1}(t)} = \sum_{k \geq 1}^{N} C_k(t)\,(\log|t|)^k \end{cases}$$

we have that $C'_k(0) = \partial C_k(0)$ for each $k = 1, \dots, N$.
Indeed, if $\varphi \in {}_c E^{n-p-1, n-q-1}_X(\text{null } D_3)$ we have that

$$(\partial\omega \wedge \delta_{T^{-1}(t)})(\varphi) = (\omega \wedge \delta_{T^{-1}(t)})((-1)^{p+q+1}\partial\varphi)$$
$$= \sum C_k(t)((-1)^{p+q+1}\partial\varphi)(\log|t|)^k = \sum (\partial C_k(t))(\varphi)\,(\log|t|)^k.$$

16| Similarly, we have that $C_{k,\bar\partial_w}(0) = \bar\partial C_{k,w}(0)$.
Therefore, since $\tilde{g}_Z \in GE^p_{|Z|}(X)$ we have that $\partial\bar\partial g$ is smooth on X, and this implies that $\partial\bar\partial C_k(0)$ is also smooth on X. This allows us to define a specialization map of Green forms:

$$\sigma_G : GE^p_{|Z|}(X) \longrightarrow GC^p_{|C_{Y\cap X} Z|}(N_Y X) \cong GE^p_{|C_{Y\cap X} Z|}(N_Y X)$$
$$\tilde{g}_Z \longmapsto \widetilde{C_0(0)}$$

Crucially, this map is compatible with the specialization map in Deligne cohomology.

$\boxed{\text{Theorem}}$

$$\begin{tikzcd}
GE^p_{|Z|}(X) \arrow[r, "cl"] \arrow[d, "\sigma_G"] & H^{2p}_{\mathcal{D},Z}(X, \mathbb{R}(p)) \arrow[d, "\sigma_{\mathcal{D}}"] \\
GC^p_{|C_{Y\cap X} Z|}(N_Y X) \arrow[r, "cl"] & H^{2p}_{\mathcal{D}, C_{Y\cap X} Z}(N_Y X, \mathbb{R}(p))
\end{tikzcd}$$

$\boxed{\text{Proof}}$ Let us recall how one can define the specialization map in Deligne cohomology:

$$\begin{tikzcd}
H^{2p}_{\mathcal{D},Z}(X, \mathbb{R}(p)) \arrow[r, dashed, "\sigma_{\mathcal{D}}"] \arrow[d, "p^*"] & H^{2p}_{\mathcal{D}, C_{Y\cap X}(Z)}(N_Y X, \mathbb{R}(p)) \\
H^{2p}_{\mathcal{D}, Z\times \mathbb{P}^1\setminus\{0\}}(X\times(\mathbb{P}^1\setminus\{0\}), \mathbb{R}(p)) \arrow[dr, "(\cdot)\cup -\frac{1}{2}\log|t|^2"', "\sim"] & H^{2p+2}_{\mathcal{D}, C_{Y\cap X}(Z)}(W^0, \mathbb{R}(p+1)) \arrow[u, "i_*", "\wr"'] \\
& H^{2p+1}_{\mathcal{D}, Z\times\mathbb{P}^1\setminus\{0\}}(X\times\mathbb{P}^1\setminus\{0\}, \mathbb{R}(p+1)) \arrow[u, "\text{boundary map}"']
\end{tikzcd}$$

which implies that, if $\tilde{g}_z$ is a Green form on X for $|z|$ then we have that:

$$\sigma_0\left(cl(\tilde{g}_z)\right) = \Big\{ -\frac{1}{4\pi i} \underbrace{\mathrm{Res}_{T^{-1}(0)}}_{\text{inside } W^{\circ} \setminus \overline{Z \times \mathbb{C}^{\times}}} \left(\log|t|^2 \cup p^*(g_z)\right) \Big\}.$$

Using the explicit description of the cup product in Deligne cohomology provided in Burgos's thesis we get:

$$\{\log|t|^2\} \cup \{p^*(g_z)\} = \Big\{ \log|t|^2 \left(\partial p^* g_z - \bar{\partial} p^* g_z\right) - \left(\partial \log|t|^2 - \bar{\partial} \log|t|^2\right) p^*(g_z) \Big\}$$

and finally we have that

$$\left(\mathrm{Res}_{T^{-1}(0)}\left(\log|t|^2 \cup p^*(g_z)\right)\right)^{(\varphi)} = \lim_{\varepsilon \to 0} \int_{\underbrace{T^{-1}(\{t \in \mathbb{C} : |t| = \varepsilon\})}_{=: \gamma_\varepsilon}} \left(\log|t|^2 \cup p^* g_z\right)(\varphi)$$

$$= \lim_{\varepsilon \to 0} \int_{\gamma_\varepsilon} \log|t|^2 \left(\partial p^* g_z - \bar{\partial} p^* g_z\right) - \int_{\gamma_\varepsilon} \left(\partial \log|t|^2 - \bar{\partial} \log|t|^2\right) p^*(g_z)$$

$$= i \lim_{\varepsilon \to 0} \log|\varepsilon|^2 \sum_{k=0}^{N} (\log|\varepsilon|)^k \int_0^{2\pi} \zeta_k'(\varepsilon e^{i\theta})(\varphi)\, d\theta + 2i \sum_{k=0}^{N} (\log|\varepsilon|)^k \int_0^{2\pi} \zeta_k(\varepsilon e^{i\theta})(\varphi)\, d\theta$$

$$= -4\pi i\, \zeta_0(0)$$

$$\Rightarrow \sigma\left(cl(\tilde{g}_z)\right) = cl\left(\widetilde{\zeta_0(0)}\right).$$

18 | As a corollary, we see that the specialization to the normal bundle yields maps

$$\sigma_G : GE^p_Z X \longrightarrow GC^p_{C_{Y \cap Z}\, Z}(N_Y X)$$

associated to each cycle $Z \in Z^p(X)$. This allows us to define maps

$$\sigma : \hat{Z}^p(X) \longrightarrow \hat{Z}^p(N_Y X)$$
$$(\{Z\}, \hat{g}_Z) \longmapsto (\{C_{Y\cap Z} Z\}, \sigma_G(g_Z))$$

and we will see next time that these pass to rational equivalence. Therefore, they define maps

$$\sigma : \widehat{CH}^p(X) \to \widehat{CH}^p(N_Y X)$$

and we will see next time that $\mathrm{Im}(\sigma) \subseteq \widehat{CH}^p(N_Y X)_{vert}$. This finally allows us to define the pullback

$$i^* : \widehat{CH}^p(X) \longrightarrow \widehat{CH}^p(N_Y X)_{vert} \underset{s^*}{\overset{\pi^*}{\underset{\longrightarrow}{\overset{\sim}{\longleftarrow}}}} \widehat{CH}^p(Y)$$

which, as we will see next time, coincides with the pullback defined by Gillet and Soulé / Burgos using the moving lemma on the generic fiber of X.