

TALK 1 Aim: If $X \hookrightarrow Y$ is a regular embedding of arithmetic varieties, define $i^*: \hat{CH}^*(Y) \rightarrow \hat{CH}^*(X)$ using the deformation to the normal cone.

To do so, we are going to use a version of $\hat{CH}^*(X)$ that works better when X is only quasi-projective, introduced by Burgos.

Ex. If X, Y are projective arithmetic varieties, $X \hookrightarrow Y$ is a regular embedding then $N_{X/Y}$ is only quasi-projective.

Ex. If k is a no field, A/k is an abelian variety the Néron model $A/\mathbb{Q}_k$ is only quasi-projective.

Ex. Moduli spaces and universal families of abelian varieties (e.g. Kuga-Sato varieties) are only quasi-projective.

$X/\mathbb{C}$ smooth variety, $\Lambda \in \mathbb{R}$ abring
 $\Rightarrow H^*(X, \mathbb{C})$ has a Hodge filtration and a Λ -structure ($H^*(X, \mathbb{C}) = H^*(X, \Lambda) \otimes_{\Lambda} \mathbb{C}$)
 $\Rightarrow$ If $\Lambda(p) := (2\pi i)^p \Lambda$ we have a map
$$H^*(X, \Lambda(p)) \rightarrow H^*(X, \mathbb{C}) / F^p H^*(X, \mathbb{C})$$

2] whose kernel and cokernel can be computed using Deligne-Beilinson cohomology

This was defined by Beilinson as follows: if $\bar{X}$ is a smooth compactification of X such that $\bar{X} \setminus X = D$ is a divisor with normal crossings, we have the constant sheaf $\mathbb{Z}(p)$ on X , as well as the $\mathbb{Z}$ -sheaf Ω_X^* of holomorphic forms and the $\mathbb{Z}$ -sheaf $\Omega_{\bar{X}}^*(\log D)$ of holomorphic forms on $\bar{X}$ having logarithmic singularities along D , with the Hodge filtration $F^p \Omega_{\bar{X}}^*(\log D) = \bigoplus_{p' \geq p} \Omega_{\bar{X}}^{p'}(\log D)$ and natural maps $v_1: Rj_* \mathbb{Z}(p) \rightarrow Rj_* \Omega_X^* = j_* \Omega_X^*$ (because $j: X \hookrightarrow \bar{X}$ is affine) and $v_2: \Omega_{\bar{X}}^*(\log D) \rightarrow j_* \Omega_X^*$. These maps can be used to define the Deligne-Beilinson complex on $\bar{X}$:

$$\mathbb{A}(p)_D := s \left(Rj_* \mathbb{Z}(p) \oplus F^p \Omega_{\bar{X}}^*(\log D) \rightarrow j_* \Omega_X^* \right) \\ \alpha \oplus \beta \mapsto v_2(\beta) - v_1(\alpha)$$

where $s(A^* \xrightarrow{u} B^*)$ denotes the mapping cone, or cone, of a map of complexes, defined as

$$s(A^* \xrightarrow{u} B^*)^n := A^n \oplus B^{n-1}$$

with differential $d(a, b) = (da, ua - db)$.

Then, the Deligne - Beilinson cohomology [3] groups of X are defined as

$$H_D^n(X, \Lambda(p)) := H^n(\bar{X}, N(p)_D)$$

and are independent of the compactification of X . To prove this, Deligne asserts that if $\bar{X}^*$ and $\bar{X}'$ are smooth compactifications of X , with $\bar{X} \setminus X$ and $\bar{X}' \setminus X$ divisors with simple normal crossings, there exists a third good compactification

$\bar{X}''$ such that $\bar{X}''$ dominates $\bar{X}$ and $\bar{X}'$. In other words, the ^{opposite of the} category of good compactifications of X is directed. Indeed, one can take as

$\bar{X}''$ a resolution of singularities of the closure of the image of X inside $\bar{X} \times \bar{X}'$.

This opportunity of varying the compactification has been used by Burgos to define a complex which computes DB-cohomology without the need of hypercohomology.

This can be done only when $\Lambda = \mathbb{R}$. We let E_X^* denote the complex of C^∞ -differentiated forms on X . Moreover, if $\bar{X}$ is a good compactification of X , and $D = \bar{X} \setminus X$, we let $E_{\bar{X}}^*(\log D)$ to be the $E_{\bar{X}}^*$ algebra of $j_X^* E_X^*$ generated ^{locally} by $\log(z_i \bar{z}_i)$, $\frac{dz_i}{d\bar{z}_i}$, $\frac{d\bar{z}_i}{z_i}$ for any local equation

4 $z_1 \dots z_n = 0$ of D . We let $E_{\bar{X}}^*(\log D)$ denote the global sections of $E_{\bar{X}}^*(\log D)$ and $E_{\bar{X}, \mathbb{R}}^*(\log D)$ be the complex of real forms. Furthermore, if X is a smooth complex manifold we define $E_{\log}^*(X) = \varinjlim_{(\bar{X}, D) \in \text{Comp}(X)^{\text{op}}} E_{\bar{X}}^*(\log D) \subseteq E_X^*$, and analogously $E_{\log, \mathbb{R}}^*(X)$.

Doing this makes it easier to detect the functoriality of forms with logarithmic poles at infinity. Now, we have the Hodge filtration

$$F^p E_{\bar{X}}^{*, n}(\log D) := \bigoplus_{\substack{p' \geq p \\ p' + q' = n}} E_{\bar{X}}^{p', q'}(\log D), \text{ and for each}$$

map $f := (\bar{X}, D) \rightarrow (\bar{X}', D')$ in $\text{Comp}(X)$ the pullback $f^*: E_{\bar{X}'}^*(\log D') \rightarrow E_{\bar{X}}^*(\log D)$ is a

filtered quasi-isomorphism. In particular, $H^*(E_{\bar{X}}^*(\log D))$ is independent of $(\bar{X}, D)$

and computes $H^*(X, \mathbb{C})$ with the Hodge

filtration. The weight filtration can also be computed by assigning weight zero to things smooth on $\bar{X}$, and weight one to the others ("filtration by the pole order").

Now, the complexes of sheaves $E_{\log, \mathbb{R}}^*(X, p)$, $E_{\log}^*(X)$ and $F^p E_{\log}^*(X)$ are acyclic, and

furthermore the natural maps:

$$Rj_* \mathbb{R}_X(p) \rightarrow E_{\log, \mathbb{R}}^*(X_{\text{up}})$$

$$j_* \Omega_X^* \rightarrow E_{\log}^*(X)$$

$$F^p \Omega_X^*(\log D) \rightarrow F^p E_{\log}^*(X)$$

are quasi-isomorphisms. Here, if we define

$$E_{\log, \mathbb{R}}^*(X_{\text{up}})_D := \left\{ \begin{array}{l} S(E_{\log, \mathbb{R}}^*(X_{\text{up}}) \oplus F^p E_{\log}^*(X) \rightarrow E_{\log}^*(X) \\ (x, \beta) \mapsto \beta \end{array} \right.$$

then we have that

$$H^n(E_{\log, \mathbb{R}}^*(X_{\text{up}})_D) \cong H_D^n(X, \mathbb{R}(p))$$

and, in fact, we could have used complexes of real-analytic differential forms throughout.

A simpler complex

Using a bit of homological algebra, we can build a simpler complex computing Deligne-Beilinson cohomology.

To do so, we start by recalling that if

$$0 \rightarrow A^* \rightarrow B^* \rightarrow C^* \rightarrow 0$$

is an exact sequence of complexes, we can define the connecting maps in cohomology

6] by taking a section $\sigma: C^* \rightarrow B^*$ of π and setting $\text{Res}_\sigma: C^*[-1] \rightarrow A^*$ to be $\text{Res}_\sigma(a) = i^{-1}(\sigma da - d\sigma a)$. If σ' is another section then Res_σ and $\text{Res}_{\sigma'}$ are homotopically equivalent, and both induce

$$H^{n-1}(C^*) \rightarrow H^n(\mathcal{S}(B^* \rightarrow C^*)) \cong H^n(A)$$

in cohomology. If $\tau: B^* \rightarrow A^*$ is a retraction of i ~~associated~~ associated to σ then

$$\text{Res}_\sigma(c) = \text{Res}_\tau(c) = d\tau b - \tau db, \quad \forall b \in B^*: \pi b = c.$$

Now, if $A^* \xrightarrow{v} B^*$ is any map of complexes, we can write

$$0 \rightarrow \ker(v) \rightarrow A^* \rightarrow \text{Im}(v) \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \text{Im}(v) \rightarrow B^* \rightarrow \text{coker}(v) \rightarrow 0$$

and if we choose a ~~retraction~~ retraction $\tau_1: \text{Im}(v) \rightarrow \ker(v)$ and a

~~section~~ section $\sigma_2: \text{coker}(v) \rightarrow B^*$ we can

define $\hat{\mathcal{S}}(v)^* := \ker(v)^* \oplus \text{coker}(v)^*[-1]$ with

$$\text{differential } d(a, b) = (da + \text{Res}_2 \text{Res}_1 b, -db)$$

with $\text{Res}_1 = \text{Res}_{\tau_1}$ and $\text{Res}_2 = \text{Res}_{\sigma_2}$. Then,

$\hat{\mathcal{S}}(v)^*$ is homotopically equivalent to $\mathcal{S}(v)$, and the maps are all explicit.

$$\varphi: \hat{\mathcal{S}}(v)^* \rightarrow \mathcal{S}(v)^*, (a, b) \mapsto (j(a) - \sigma_2 \text{Res}_1(b), \sigma_1 b)$$

$$\psi: \mathcal{S}(v)^* \rightarrow \hat{\mathcal{S}}(v)^*, (a, b) \mapsto (\tau_1 a + \text{Res}_2(\tau_1 b), \tau_1 b)$$

$$\psi \varphi = \text{Id}, \quad \varphi \psi - \text{Id} = dh + hd, \quad \text{where}$$

$$h: \mathcal{S}(v)^* \rightarrow \mathcal{S}(v)^*[-1], (a, b) \mapsto (-\sigma_2 \tau_1 b, 0)$$

We want to take $A^* = E_{\log, \mathbb{R}}^*(X, p) \oplus F^p E_{\log}^*(X)$ [7]
 and $B^* := E_{\log}^*(X)$, where $v(a, b) = b - a$. Then, it
 turns out that v is injective for $* \leq 2p-1$ and
 surjective for $* \geq 2p-1$, while in general
 $\text{coker}(v)^n =$ ~~$E_{\log, \mathbb{R}}^n(X, p) \oplus F^p E_{\log}^n(X)$~~
 $E_{\log}^n(X) / E_{\log, \mathbb{R}}^n(X, p) + F^p E_{\log}^n(X) + \bar{F}^p E_{\log}^n(X)$

$$\begin{aligned} &\cong E_{\log, \mathbb{R}}^n(X, p-1) / E_{\log, \mathbb{R}}^n(p-1) \cap (F^p E_{\log}^n(X) + \bar{F}^p E_{\log}^n(X)) \\ &\cong E_{\log, \mathbb{R}}^n(X, p-1) \cap \bigoplus_{\substack{p'+q'=n \\ p' < p, q' < p}} E_{\log}^{p', q'}(X) \end{aligned}$$

$$\begin{aligned} \text{ker}(v)^n &\cong E_{\log, \mathbb{R}}^n(X, p) \cap F^p E_{\log}^n(X) \cap \bar{F}^p E_{\log}^n(X) \\ &\cong E_{\log, \mathbb{R}}^n(X, p) \cap \bigoplus_{\substack{p'+q'=n \\ p' \geq p, q' \geq p}} E_{\log}^{p', q'}(X) \end{aligned}$$

With these descriptions we can easily
 find a section of $\pi: E_{\log}^*(X) \rightarrow \text{coker}(v)^*$,
 by ~~taking~~ taking the natural inclusion
 $E_{\log, \mathbb{R}}^n(X, p-1) \hookrightarrow E_{\log}^n(X)$, and a

2 retraction ~~of~~ of $j: \ker(u) \hookrightarrow E_{\log, \mathbb{R}}^*(X, p) \oplus F^p E_{\log}^*(X)$
 by setting $\tau(\alpha, \beta) = \sum_{\substack{p' \geq p \\ q' \geq p}} \alpha^{p', q'}$. This yields the

complex $D^*(E_{\log}^*(X), p)$ given by:

$$D^n(E_{\log}^*(X), p) = \begin{cases} E_{\log, \mathbb{R}}^{n-1}(X, p-1) \cap \bigoplus_{\substack{p'+q'=n-1 \\ p' \geq p, q' \leq p}} E_{\log}^{p', q'}(X), & \text{if } n \leq 2p-1 \\ E_{\log, \mathbb{R}}^n(X, p) \cap \bigoplus_{\substack{p'+q'=n \\ p' \geq p, q' \geq p}} E_{\log}^{p', q'}(X), & \text{if } n \geq 2p \end{cases}$$

with differential given by

$$d(\alpha) = \begin{cases} -\pi_{p-1}^{n-p+1, n-p+1}(F^{\bullet}(d\alpha)), & \text{for } n \leq 2p-2 \\ -2\partial\bar{\partial}\alpha, & \text{for } n = 2p-1 \\ d\alpha, & \text{for } n \geq 2p \end{cases}$$

where $\pi_{p-1}^{n-p+1, n-p+1}(F^{\bullet}(\beta)) = \sum_{\substack{p' \geq n-p+1 \\ q' \geq n-p+1}} \beta^{p', q'}$ while $\pi_{p-1}(\sigma) = \frac{1}{2}(\sigma + (-1)^{p-1}\bar{\sigma})$

As we have seen before, $D^*(E_{\log}^*(X), p)$ is homotopically equivalent to $E_{\log, \mathbb{R}}^*(X, p)$.

Therefore $H_{\mathbb{D}}^n(X, \mathbb{R}(p)) \cong H^n(D^*(E_{\log}^*(X), p))$. 9

In particular, we see that

$$H_{\mathbb{D}}^{2p}(X, \mathbb{R}(p)) = \left\{ \alpha \in E_{\log}^{p,p}(X) \cap E_{\log, \mathbb{R}}^{2p}(X, p) \mid d\alpha = 0 \right\} / \frac{\text{Im}(\partial) + \text{Im}(\bar{\partial})}{\text{Im}(\partial)}$$

$$H_{\mathbb{D}}^{2p-1}(X, \mathbb{R}(p)) = \left\{ \alpha \in E_{\log}^{p-1,p-1}(X) \cap E_{\mathbb{R}}^{2p-1}(X, p-1) \mid \partial \bar{\partial} \alpha = 0 \right\} / \text{Im}(\partial) + \text{Im}(\bar{\partial})$$

There is also a natural product on the complex $D^*(E_{\log}^*(X), p)$, given by the maps

$$D^n(E_{\log}^*(X), p) \otimes D^m(E_{\log}^*(X), q) \rightarrow D^{n+m}(E_{\log}^*(X), p+q)$$

defined by $x \cdot y = \begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y), & \text{if } n < 2p \text{ and } m < 2q \end{cases}$

This product is graded-commutative but it is associative only up to homotopy.

Here $r_p(x) = 2r_p(F^p dx)$

whereas π is the projection onto $\text{ker}(v)$.

$$\begin{cases} \pi(x \wedge y), & \text{if } n < 2p \text{ and } m \geq 2q \\ & \text{and } n+m < 2(p+q) \end{cases}$$

$$F^{p+q, p+q}(r_p(x) \wedge y) + 2r_p(\partial((x \wedge y)^{p+q-1, n+m-p-q})),$$

$$\begin{cases} \text{if } n < 2p \text{ and } m \geq 2q \text{ and } n+m \geq 2(p+q) \end{cases}$$

$$\begin{cases} x \wedge y, & \text{if } n \geq 2p, m \geq 2q \\ (-1)^{nm} y \cdot x, & \text{if } n \geq 2p \text{ and } m < 2q. \end{cases}$$

10 Truncated relative cohomology

$f: A^* \rightarrow B^*$ fixed map of complexes

$$\hat{H}^n(A^*, B^*) = \left\{ (a, \tilde{b}) \in A^n \oplus \frac{B^{n-1}}{\text{Im } d} \mid \begin{array}{l} f(a) = db, \\ da = 0 \end{array} \right\}$$

This lies between the cohomology of B^* and the relative cohomology $H^*(A^*, B^*) = H^*(S(f))$.

Indeed, if $\sigma^p A^n := \begin{cases} A^n, & \text{if } n \geq p \\ 0, & \text{if } n < p \end{cases}$ is the stupid (bête) filtration on A^* , we get that

$$H^n(S(\sigma^p A^* \xrightarrow{f} B^*)) = \begin{cases} H^{n-1}(B^*) & \text{if } n < p \\ \hat{H}^n(A^*, B^*) & \text{if } n = p \\ H^n(A^*, B^*) & \text{if } n > p \end{cases}$$

We have various maps:

$$\iota: \hat{H}^n(A^*, B^*) \rightarrow H^n(A^*, B^*), (a, \tilde{b}) \mapsto \{(a, \tilde{b})\}$$

$$w: \hat{H}^n(A^*, B^*) \rightarrow \mathbb{Z} A^n, (a, \tilde{b}) \mapsto a$$

$$\{a \in A^n \mid da = 0\}$$

$$a: \tilde{A}^{n-1} \rightarrow \hat{H}^n(A^*, B^*), \tilde{a} \mapsto (da, \tilde{f(a)})$$

$$b: H^{n-1}(B^*) \rightarrow \hat{H}^n(A^*, B^*), \{b\} \mapsto (0, \tilde{b})$$

fitting in exact sequences

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$$\begin{aligned}
 H^{n-1}(A^*, B^*) &\rightarrow \tilde{H}^{n-1} \xrightarrow{a} \hat{H}^n(A^*, B^*) \xrightarrow{d} H^n(A^*, B^*) \rightarrow 0 \\
 0 &\rightarrow H^{n-1}(B^*) \xrightarrow{b} \hat{H}^n(A^*, B^*) \xrightarrow{\sim} ZA^n \rightarrow H^n(B^*) \\
 H^{n-1}(A^*, B^*) &\rightarrow H^{n-1}(A^*) \rightarrow \hat{H}^n(A^*, B^*) \rightarrow H^n(A^*, B^*) \oplus ZA^n \rightarrow H^n(A^*) \\
 &\rightarrow 0
 \end{aligned}$$

induced in fact from the exact sequences of complexes:

$$0 \rightarrow s(\sigma^n A^* \rightarrow B^*) \rightarrow s(A^* \rightarrow B^*) \rightarrow A^*/\sigma^n A^* \rightarrow 0$$

$$0 \rightarrow B^*[-1] \rightarrow s(\sigma^n A^* \rightarrow B^*) \rightarrow \sigma^n A^* \rightarrow 0$$

$$0 \rightarrow s(\sigma^n A^* \rightarrow B^*) \rightarrow s(A^* \rightarrow B^*) \oplus \sigma^n A^* \rightarrow A^* \rightarrow 0$$

There is also a purely algebraic way to get a product on truncated relative cohomology groups. Namely, take $A^* \xrightarrow{f} B^*$ and $C^* \xrightarrow{g} D^*$, build

$$\begin{array}{ccc}
 A^* \otimes C^* & \xrightarrow{1 \otimes g} & A^* \otimes D^* \\
 \downarrow f \otimes 1 & & \downarrow f \otimes 1 \\
 B^* \otimes C^* & \xrightarrow{1 \otimes g} & B^* \otimes D^*
 \end{array} \xrightarrow{\times} \begin{pmatrix} E_1^* & \rightarrow & E_2^* \\ \downarrow & & \downarrow \\ E_3^* & \rightarrow & E_4^* \end{pmatrix}$$

and observe that we get an induced product:

$$\begin{aligned}
 s(f) \otimes s(g) &\xrightarrow{\sim} s(A^* \otimes C^* \rightarrow s(A^* \otimes D^*) \oplus (B^* \otimes C^*) \rightarrow B^* \otimes D^*) \\
 &\downarrow \\
 &s(E_1^* \rightarrow s(E_2^* \oplus E_3^*) \rightarrow E_4^*)
 \end{aligned}$$

12 which yields the following product:

$$H^n(A^*, B^*) \otimes H^m(C^*, D^*) \rightarrow H^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \rightarrow E_4^*))$$

$$\{(a, b) \otimes (c, d)\} \mapsto (ac, bc + (-1)^n ad, (-1)^n bd)$$

This product can be lifted to:

$$\hat{H}^n(A^*, B^*) \otimes \hat{H}^m(C^*, D^*) \rightarrow \hat{H}^{n+m}(E_1^*, s(E_2^* \oplus E_3^* \rightarrow E_4^*))$$

$$(a, \tilde{b}) \otimes (c, \tilde{d}) \mapsto (a\tilde{c}, (bc + (-1)^n ad, (-1)^n bd) \sim)$$

which is well-defined and compatible with w and cl .

Green ~~forms~~ $X/\mathbb{C}$ smooth algebraic

$Z \in X$ ^{alg} subvariety of codimension p

$$\Rightarrow GE_Z^p(X) = \hat{H}^{2p}(D^*(E_{\log}^*(X), p), D^*(E_{\log}^*(X, Z), p))$$

$$(w, \tilde{g}) \in GE_Z^p(X) \Rightarrow w \in Z D^{2p}(E_{\log}^*(X), p)$$

$$\Downarrow \quad = \left\{ w \in E_{\log}^{p,p}(X) \cap E_{\log, \mathbb{R}}^{2p}(X, p) \mid dw = 0 \right\}$$

$$\tilde{g} \in \bar{D}^{2p-1}(E_{\log}^*(X, Z), p)$$

$$= \left(E_{\log}^{p-1, p-1}(X, Z) \cap E_{\log, \mathbb{R}}^{2p-2}(X, Z, p-1) \right) / \ln \partial + \ln \bar{\partial}$$

$$=: \tilde{E}_{\log, \mathbb{R}}^{p-1, p-1}(X, Z)$$

Moreover, $w = -2 \partial \bar{\partial} g$. Therefore, we have ^(smooth on X)

$$GE_Z^p(X) \cong \left\{ \tilde{g} \in \tilde{E}_{\log, \mathbb{R}}^{p-1, p-1}(X, Z) \mid \partial \bar{\partial} g \in E_{\log, \mathbb{R}}^{p,p}(X) \right\}$$

$Z^p(X) := \{ \text{codimension } p \text{ alg. subsets of } X \}$

$GE^p_Z(X) := \varinjlim_{Z \in Z^p(X)} GE^p_Z(X)$ is the space of Green forms with singularities in codimension at least p .

From the general homological algebra we get:

$$0 \rightarrow \tilde{E}_{\log, \mathbb{R}}^{p-1, p-1}(X) \xrightarrow{\alpha} GE^p_Z(X) \xrightarrow{d} H_{D, Z}^{2p}(X, \mathbb{R}(p)) \rightarrow 0$$

$$0 \rightarrow H_D^{2p-1}(X, \mathbb{Z}, \mathbb{R}(p)) \rightarrow GE^p_Z(X) \rightarrow \tilde{E}_{\log, \mathbb{R}}^{p, p}(X) \rightarrow H_D^{2p}(X, \mathbb{Z}, \mathbb{R}(p))$$

$$0 \rightarrow H_D^{2p-1}(X, \mathbb{R}(p)) \xrightarrow{\alpha} GE^p_Z(X) \rightarrow H_{D, Z}^{2p}(X, \mathbb{R}(p)) \oplus \tilde{E}_{\log, \mathbb{R}}^{p, p}(X) \rightarrow H_D^{2p}(X, \mathbb{R}(p)) \rightarrow 0$$

because $H_{D, Z}^{2p-1}(X, \mathbb{R}(p)) = 0$, which follows from

the fact that $H_Z^n(X, \mathbb{R}) = H_Z^n(X, \mathbb{C}) = 0, \forall n < 2p$.

If $Z \in Z^p(X)$ then we define $\tilde{p}$ a codimension p alg. cycle in $H_{D, Z}^{2p}(X, \mathbb{R}(p))$

$$GE^p_Z(X) := \left\{ \tilde{g} \in GE^p_{|Z|}(X) \mid d(\tilde{g}) = \tilde{p}(Z) \right\}$$

where the cycle class map p is defined by the fact that the image of $[Z]$ via $CH^p(X) \rightarrow H^{2p}(X, \mathbb{C})$ lies in $F^p H^{2p}_Z(X, \mathbb{C})$.

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Then, we have that

$$GE^p_2(X) = \left\{ g \in E^{p-1,p-1}_{\log, \mathbb{R}}(X - \mathbb{Z}, p-1) \mid \begin{array}{l} \partial \bar{\partial} g \in E^{2p}_X \\ \{-2\partial \bar{\partial} g, g\} = p(Z) \end{array} \right\}$$

$$\operatorname{Im} \partial + \operatorname{Im} \bar{\partial}$$

and we also have an isomorphism $H^{2p}_{D, \mathbb{R}}(X, \mathbb{R}(p)) \xrightarrow{\sim} H^{2p}_V(X, \mathbb{R})$
induced by the map of complexes

$$\begin{array}{c} \frac{1}{(2\pi i)^p} r_p : s(D^*(E^*_{X, p}), D^*(E^*_{\log}(X - \mathbb{Z}), p)) \\ \rightarrow s(E^*_{X, \mathbb{R}} \rightarrow E^*_{\log, \mathbb{R}}(X - \mathbb{Z})) \end{array}$$

such that $\left(\frac{1}{(2\pi i)^p} r_p\right)(w, g) = \frac{1}{(2\pi i)^p}(w, 2d^c g)$ in degree $2p$. Therefore, we see through this map that $GE^p_2(X)$ can be identified with

$$GE^p_2(X)' = \left\{ g \in E^{p-1,p-1}_{\log, \mathbb{R}}(X, \mathbb{R}) \mid \begin{array}{l} \det^c g \in E^{p,p}_X \\ \{d d^c g, d g\} = \{Z\} \end{array} \right\}$$

$$\operatorname{Im} \partial + \operatorname{Im} \bar{\partial}$$

where now the cohomology classes are taken in $H^{2p}_{\mathbb{Z}}(X, \mathbb{R})$.

This last space can be identified with 15
 the space of ~~Green~~ currents for $\mathbb{C}$. To do so,
 Burgos proceeds through several steps:

- first of all, Burgos shows that there is a map $E_X^*(\log D) \rightarrow D_{X|D}^*$ for every smooth projective X and every $D \subseteq X$ with normal crossings, where $D_{X|D}^*$ is the topological dual of the differential forms that vanish along D .

The aforementioned map is then given by $g \mapsto [g]$, with $[g](\varphi) = \int_X g \wedge \varphi$.

To do so, one reduces to an integral over a polydisk $X = \Delta^n$ with $D = \{z_1 = \dots = z_r = 0\}$

and one gets $|\int_X g \wedge \varphi| \ll \int_{\Delta^n} \frac{\prod_{i=1}^r |\log(r_i)|^n}{\prod_{i=1}^r r_i} dV$

where the integral on the right converges. The bound comes from the estimate $|\varphi| \leq 2^r \cdot \sup_{\substack{z \in \Delta^n \\ \lim_{r \rightarrow 0} \tilde{r}_r = 0 \\ 1 \leq r \leq r}} \left| \frac{\partial^r \varphi}{\partial z_1 \dots \partial z_r \partial \bar{z}_1 \dots \partial \bar{z}_r} \right|_z$

One shows similarly that the functional $\varphi \mapsto \int_X g \wedge \varphi$ is continuous, and thus

16 defines a current. Finally, we have $d[g] = [dg]$.

- one can analogously define $[\cdot]: E_X(\log Y) \rightarrow D_{X/Y}$ if $Y \hookrightarrow X$ is a codimension p subvariety;
- if $\tilde{g} \in GE_Z^p(X)'$, then $\tilde{g}$ can be written, on a good compactification $(\bar{X}, D)$ of $X \setminus Z$, as $g = \sum \alpha_i \lambda_i + \beta$ where λ_i, λ_r are Green forms for D , while α_i and β are smooth differential forms on $\bar{X}$, such that $\alpha_i|_{D_i}$ is ∂ and $\bar{\partial}$ closed. (compare Griffiths-Schmid). These representatives are called basic Green forms.

- Now, $d(g) = \{ (dd^c g, d^c g) \} \in H_{\mathbb{Z}}^{2p}(X, \mathbb{C})$ equals the class $\{ \lambda \}$ by definition of $GE_Z^p(X)'$, and on the other hand $\text{Res}(d^c g) = c \bar{\partial}_Z$ for some constant c . Therefore $c=1$, and $dd^c(g) + \bar{\partial}_Z = dd^c g$.

Let us use this definition of Green forms 17
 to define arithmetic Chow groups. First of all
 one sets $\widehat{Z}^p(X) = \left\{ (Z, g) \mid \begin{array}{l} Z \in Z^p(X) \\ g \in GE_{\mathbb{Z}}^p(X, \mathbb{R}) \end{array} \right\}$

for any arithmetic variety X . Then, we can
 define rational equivalence by looking at
 arithmetic cycles of the form $\widehat{\text{div}}(f) = (\text{div } f, -b(p(f)))$ where $b: H^{2p-1}((X - \text{div}(f))_{\mathbb{R}}, \mathbb{R}(p))$
 $\rightarrow GE_{\mathbb{Z}}^p(X, \mathbb{R})$ while p denotes Beilinson's

regulator, and $f \in K(W)^*$ for some $W \in X$
 of codimension $p-1$. In this setting, the Beilinson
 regulator can be defined by taking the image
 of $\frac{1}{2} \log(f/\bar{f}) \in D_{\text{log}}'(E_{\text{log}}(W - \text{div}(f)), 1)$

under the Gysin morphism induced by
 the inclusion $W \in X$. Then, we can
 define $\widehat{\text{Rat}}^p(X) := \langle \widehat{\text{div}}(f) \mid W \in X, \text{codim } W = p-1 \rangle$

Then, one defines $\widehat{\text{CH}}^p(X, D(E_{\text{log}})) = \frac{\widehat{Z}^p(X)}{\widehat{\text{Rat}}^p(X)}$
 and Burgos shows that these groups

18 coincide with those defined by Deligne and Serre if X_K/k is proper.

More precisely, $\widehat{CH}^p(X, \mathcal{O}(\bar{E}_{\log})) \cong \widehat{CH}^p(X)$
 $(z, \tilde{g}) \mapsto (z, z(2\pi i)^{d_p + 1} [\tilde{g}])$

There are several maps:

$$z: \widehat{CH}^p(X) \rightarrow CH^p(X), z(\{z, \tilde{g}\}) = \{z\}$$

$$p: CH^p(X) \rightarrow H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)) \text{ cycle class}$$

$$\alpha: \tilde{E}_{\log}^{p,p-1}(X_{\mathbb{R}}) \rightarrow \widehat{CH}^p(X), \alpha(\tilde{g}) = (0, \tilde{g})$$

$$w: \widehat{CH}^p(X) \rightarrow \mathbb{Z} E_{\log}^{p,p}(X_{\mathbb{R}}), w(z, \tilde{g}) = -2\partial\bar{\partial}\tilde{g}$$

$$h: \mathbb{Z} E_{\log}^{p,p}(X_{\mathbb{R}}) \rightarrow H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)), h(\alpha) = \{-\}$$

and we can define groups

$$CH^p(X)_0 := \ker(p), \quad \widehat{CH}^p(X)_0 := \ker(w).$$

We have a commutative diagram:

$$\begin{array}{ccc} \widehat{CH}^p(X) & \xrightarrow{w} & \mathbb{Z} E_{\log}^{p,p}(X_{\mathbb{R}}) \\ \downarrow z & \parallel & \downarrow h \\ CH^p(X) & \xrightarrow{p} & H_{\mathcal{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)) \end{array}$$

as well as exact sequences

$$CH^{p,p-1}(X) \xrightarrow{\mathcal{P}} \tilde{E}_{\log}^{p-1,p-1}(X_{\mathbb{R}}) \cong \hat{CH}^p(X) \xrightarrow{\mathcal{Z}} CH^p(X) \rightarrow 0$$

$$CH^{p,p-1}(X) \xrightarrow{\mathcal{P}} H_{\mathbb{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \cong \hat{CH}^p(X)$$

$$\xrightarrow{(Z, -w)} CH^p(X) \oplus \mathbb{Z} E_{\log}^{p,p}(X_{\mathbb{R}})$$

$$\xrightarrow{\mathcal{P} \mapsto h} H_{\mathbb{D}}^{2p}(X_{\mathbb{R}}, \mathbb{R}(p)) \rightarrow 0$$

$$CH^{p,p-1}(X) \xrightarrow{\mathcal{P}} H_{\mathbb{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) \cong \hat{CH}^p(X)_0$$

$$\xrightarrow{\mathcal{Z}} CH^p(X)_0 \rightarrow 0$$

We can use these exact sequences to compute the arithmetic Chow group of a vector bundle. More precisely, let $\pi: E \rightarrow X$ be a vector bundle, and let $\hat{CH}^p(E)_{\text{vert}} := \{(Z, g) \in \hat{CH}^p(X) \mid -\mathbb{Z} \partial \bar{\partial} g = \pi^*(w), w \in E_{\log}^{p,p}(X_{\mathbb{R}})\}$. Then $\pi^*: \hat{CH}^p(X) \rightarrow \hat{CH}^p(E)$ is injective and its image is $\hat{CH}^p(E)_{\text{vert}}$. Indeed, $\text{Im}(\pi^*) \in \hat{CH}^p(E)_{\text{vert}}$ clearly. Moreover we have the diagram

$$\begin{array}{ccccccc} CH^{p,p-1}(X) & \xrightarrow{\mathcal{P}} & H_{\mathbb{D}}^{2p-1}(X_{\mathbb{R}}, \mathbb{R}(p)) & \rightarrow & \hat{CH}^p(X) & \rightarrow & CH^p(X) \oplus \mathbb{Z} E_{\log}^{p,p}(X_{\mathbb{R}}) \\ \downarrow \pi^* & & \downarrow & & \downarrow & & \downarrow \pi^* \\ CH^{p,p-1}(E) & \xrightarrow{\mathcal{P}} & H_{\mathbb{D}}^{2p-1}(E_{\mathbb{R}}, \mathbb{R}(p)) & \rightarrow & \hat{CH}^p(E)_{\text{vert}} & \rightarrow & CH^p(E) \oplus \pi^* \mathbb{Z} E_{\log}^{p,p}(E_{\mathbb{R}}) \end{array}$$

which allows us to conclude by the five lemma. To conclude, we recall that the $\star$ -product of Green forms involves the

20 | ~~the~~ product of cycles intersecting properly. What happens for nonproper intersections??