NOETHER–LEFSCHETZ DIVISORS ARE THE COEFFICIENTS OF A MODULAR FORM

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Abstract. Given a pencil of quartics, how many of the smooth fibers contain a curve of given degree and genus? The solutions to these problems for all degree and genus form the coefficients of an explicit modular form. Following Maulik and Pandharipande (2013) we outline the proof of this statement and state the modular form. The main technical result behind the proof is a theorem of Borcherds on modularity of Heegner divisors. We also state Borcherds' theorem and summarize its proof. This theorem applies not just to Noether–Lefschetz divisors on moduli of polarized K3 surfaces but to Heegner divisors in moduli spaces of hyperkähler varieties.

1. Introduction

These are the lecture notes for my talk in the Bonn–Paris seminar for the "Moduli spaces of K3 surfaces and hyperkähler manifolds." Although the long term goal of our seminar is to understand the cohomology of moduli of hyperkähler varieties, this talk will revolve around the solution to the following motivating problem.

Motivating problem. Given a generic pencil of quartic surfaces in $\mathbb{P}^3$, how many of the smooth quartics in the pencil contain a curve of degree d and genus h?

Bear in mind that the solution to this problem introduces us to many of the ideas needed to tackle the hyperkähler case. For the first half of these notes, we will discuss the solution to this problem. In the second half of these notes, we will discuss the engine behind the proof (Borcherds' theorem 3.3), which has a broad range of applications.

The motivating problem has a classical appearance but is resistant to standard techniques. Each (h, d) poses an enumerative problem, only the first few that were previously solved. Using modern ideas, Maulik and Pandharipande [6] solve this problem for all (h, d) at once! To give a few examples:

(h,d)	$(0,1)$	$(0,2)$	$\dots$	$(2,5)$	$\dots$
$\#$	320	2008	$\dots$	136512	$\dots$

The full solution is given by the coefficients of (1.3.6), see also the expansion below it.

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1.1. **The Noether–Lefschetz divisors in moduli of polarized K3s.** Let $\mathcal{F}_\ell$ be the moduli space of quasi-polarized K3 surfaces of degree ℓ. An object in this moduli space is a pair (S, v) where S is a K3 surface and $v \in \mathrm{H}^2(S, \mathbb{Z})$ is the first Chern class of an ample line bundle L on S with $v^2 = \ell$. Recall that the Noether–Lefschetz divisors are given by

$$\text{(1.1.1)}\quad \mathrm{NL}_{h,d} = \{(S, v) \in \mathcal{F}_\ell \mid \exists u \in \mathrm{H}^2(S, \mathbb{Z})\ v\cdot u = d,\ u^2 = 2h-2\}, \quad h, d \in \mathbb{Z}.$$

Roughly speaking, the locus $\mathrm{NL}_{h,d}$ parametrizes polarized K3s (S, L) containing a genus h curve that has degree d with respect to the line bundle L.

A family $(\pi\colon \mathcal{X} \to C, \mathcal{L})$ of polarized K3s induces a map $C \to \mathcal{F}_\ell$ from the base of the family to the moduli space. If C is a complete curve and $C \to \mathcal{F}_\ell$ is generic enough then the number of fibers of π containing a curve of degree d and genus h is exactly the number of intersection points of C and $\mathrm{NL}_{h,d}$. (This is only a white lie and would be correct if stated carefully.)

The space $\mathcal{F}_\ell$ is a smooth orbifold, so Weil divisors are $\mathbb{Q}$-Cartier. In particular, each $\mathrm{NL}_{h,d}$ induces an element in $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_\ell)$. We suppress the map $C \to \mathcal{F}_\ell$ from notation and denote by $C \cdot \mathrm{NL}_{h,d}$ the degree of the pullback of $\mathrm{NL}_{h,d}$ to C (namely, the intersection number in $\mathcal{F}_\ell$).

The age old way to solve an enumerative problem such as computing $C \cdot \mathrm{NL}_{h,d}$ goes as follows:

(1) Find "simple" divisors $E_1, \ldots, E_r \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_\ell)$ that form a basis.
(2) Express $\mathrm{NL}_{h,d}$ in terms of E_i's, that is, find $a_i \in \mathbb{Q}$ such that $\mathrm{NL}_{h,d} = \sum_{i=1}^r a_i E_i$.
(3) Compute $C \cdot E_i$.

Traditionally, each of these is a geometric problem to be solved. Now, we will use the arithmetic of modular forms to solve them for all (h, d) at once.

1.2. **A non-reduced structure on the Noether–Lefschetz divisors and their modularity.** First, by making *some* of the components of $\mathrm{NL}_{h,d}$ of multiplicity two, we can get another divisor $D_{h,d}$ which is more "symmetric". This operation is invertible in the sense that knowing the intersection number of C with D's gives the intersection number for NL's, and vice versa. See the introduction of [6] for the precise definition of $D_{h,d}$'s. We emphasize that $D_{h,d}$ is supported on $\mathrm{NL}_{h,d}$. See Remark 1.4 for some intuition behind the multiplicities.

In the first half of these notes, one of our goals is to state the following theorem of Borcherds precisely (See Theorem 2.4). We will also demonstrate how it can be exploited to solve our motivating problem. In the second half, we will state it in greater generality (Theorem 3.3) and summarize its proof.

Theorem 1.1 (Borcherds)**.** *The divisor classes $D_{h,d} \in \mathrm{Pic}_{\mathbb{Q}}(\mathcal{F}_\ell)$ are the Fourier coefficients of a vector-valued modular form.*

Remark 1.2. In particular, for a complete curve $C \to \mathcal{F}_\ell$, the degrees $C \cdot D_{h,d}$ form the Fourier coefficients of an "honest" vector-valued modular form.

1.3. **Pencil of quartic surfaces, a case study.** Let us now fix a generic pencil of quartics $\mathbb{P}^1 \to |\mathcal{O}_{\mathbb{P}^3}(4)|$, which gives a map $\mathbb{P}^1 \to \mathcal{F}_4$. We now want to state the recipe by which one can determine the intersection numbers $\mathbb{P}^1 \cdot D_{h,d}$. The recipe below already suggests that modularity is at play, even though it is currently hidden.

Let $\Delta_\ell(h,d) = d^2 - \ell(2h-2)$ and define a sequence $\{a_n\}_{n\in\frac{1}{8}\mathbb{N}}$ as follows:

$$a_n = \begin{cases} \mathbb{P}^1 \cdot D_{h,d} & \text{if } n = \frac{1}{8}\Delta_4(h,d), \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.2}$$

Consider the following generating series

$$\varphi(q) = \sum_{n\in\frac{1}{8}\mathbb{N}} a_n q^n. \tag{1.3.3}$$

Theorem 1.3 (Proposition 5, Maulik–Pandharipande)**.** *The power series $\varphi(q)$ is a homogeneous polynomial of degree* 21 *in*

$$A = \sum_{n\in\mathbb{Z}} q^{n^2/8} \quad \textit{and} \quad B = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2/8}. \tag{1.3.4}$$

The first proof of this statement given by Maulik and Pandharipande is somewhat *ad hoc*. We will see later that their second proof generalizes well to other families of K3s.

To figure out exactly which polynomial in A and B equals $\varphi(q)$, we need to solve for 22 coefficients. Maulik and Pandharipande use Gromov–Witten theory to solve for Noether–Lefschetz numbers on the quartic pencil with $h = 0$, this gives them sufficiently many independent a_n's to deduce the coefficients of the polynomial in A and B.

The exact polynomial is

$$\begin{aligned} 2^{22}\varphi(q) \quad = \quad & 3A^{21} - 81A^{19}B^2 - 627A^{18}B^3 - 14436A^{17}B^4 \\ & -20007A^{16}B^5 - 169092A^{15}B^6 - 120636A^{14}B^7 \\ & -621558A^{13}B^8 - 292796A^{12}B^9 - 1038366A^{11}B^{10} \\ & -346122A^{10}B^{11} - 878388A^9B^{12} - 207186A^8B^{13} \\ & -361908A^7B^{14} - 56364A^6B^{15} - 60021A^5B^{16} \\ & -4812A^4B^{17} - 1881A^3B^{18} - 27A^2B^{19} + B^{21}. \end{aligned} \tag{1.3.5}$$

Note, however, that the fibers of the quartic pencil also contribute to the count here. But it is easy to correct this contribution. The following generating series counts only the contribution from the smooth fibers:

$$\varphi(q) - 108 \sum_{n\in\mathbb{Z}_{>0}} q^{n^2}. \tag{1.3.6}$$

Expanding out a few terms of (1.3.6), we get the following result:

```
-1 + 320*q^{9/8} + 5016*q^{3/2} + 76950*q^{2} + 136512*q^{17/8} + 640224*q^{5/2} +
    3611400*q^{3} + 5329152*q^{25/8} + 15621984*q^{7/2} + 55656396*q^{4} +
    74481216*q^{33/8} + 170398720*q^{9/2} + 462721896*q^{5} + 585601920*q^{41/8} +
    1144302120*q^{11/2} + 2620503408*q^{6} + 3184479360*q^{49/8} +
    5605653600*q^{13/2} + 11311490160*q^{7} + 13396110144*q^{57/8} +
    21786141216*q^{15/2} + 40300330950*q^{8} + 46648372608*q^{65/8} +
    71683494912*q^{17/2} + 123140741160*q^{9} + 140521096512*q^{73/8} +
    205816117320*q^{19/2} + 335690102736*q^{10} + 377376426560*q^{81/8} +
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533634091200*q^{21/2} + 828581828328*q^{11} + 923347062720*q^{89/8} +
1263920168160*q^{23/2} + 1897438866000*q^{12} + 2091731874624*q^{97/8} + ...
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Remark 1.4. Note that 320 is the number of quartics containing a line. But $5016 = 2\times 2008$ is double the number of quartics containing a conic. Morally, this is because a conic never appears alone but they appear in pairs (the plane spanned by the conic will contain yet another conic). This counting with multiplicity is more natural and it is the reason for introducing the non-reduced divisor $D_{h,d}$ supported on $\mathrm{NL}_{h,d}$.

2. A better modularity for Noether–Lefschetz numbers

We now explain the second proof of Maulik and Pandharipande. This results in a more symmetric form, allowing for simpler computations that generalize well.

2.1. **The metaplectic group.** Let $\mathrm{SL}_2(\mathbb{R}) = \{A \in \mathrm{GL}_2(\mathbb{R}) \mid \det A = 1\}$. Note that $\mathrm{SL}_2(\mathbb{R})$ agrees with the symplectic group $\mathrm{Sp}_2(\mathbb{R})$. The group $\mathrm{SL}_2(\mathbb{R})$ admits a unique unramified double cover $\mathrm{Mp}_2(\mathbb{R})$ whose elements can be represented as pairs $(A, \varepsilon_A(\tau))$ where $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ and $\varepsilon_A(\tau) = \sqrt{c\tau + d}$ is one of the roots of the linear function $c\tau + d$ on the upper half plane $\mathbb{H}$. Multiplication is given by

$$(A, \varepsilon_A)(B, \varepsilon_B) = (AB, \varepsilon_A(B\cdot\tau)\varepsilon_B(\tau)) \tag{2.1.7}$$

where $B\cdot\tau \in \mathbb{H}$ denotes the usual action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$.

The integral metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$ can be generated by two elements

$$T = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right) \quad S = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{\tau}\right), \tag{2.1.8}$$

where $\sqrt{\tau}$ denotes the square root with positive real and imaginary part. The action of $\mathrm{Mp}_2(\mathbb{Z})$ on $\mathbb{H}$ has only a single cusp and that is $i\infty$.

2.2. **Vector-valued modular forms.** For fixed $(S, v) \in \mathcal{F}_\ell$, let $\Lambda = \mathrm{H}^2(S,\mathbb{Z})_{\mathrm{prim}}$ be our fixed K3 lattice orthogonal to the polarization. Let $\Lambda^\vee \subset \Lambda \otimes_\mathbb{Z} \mathbb{Q}$ be the dual of Λ. The intersection product on Λ extends $\mathbb{Q}$-linearly to $\Lambda^\vee$, giving a $\mathbb{Q}/2\mathbb{Z}$-valued form on the discriminant group $\Lambda^\vee/\Lambda$. Note that

$$\Lambda^\vee/\Lambda \simeq \mathbb{Z}/\ell\mathbb{Z}. \tag{2.2.9}$$

Our modular forms will take values in the group ring of $\Lambda^\vee/\Lambda$, that is, in

$$V \overset{\mathrm{def}}{=} \mathbb{C}[\Lambda^\vee/\Lambda] = \mathbb{C}\langle v_0, v_1, \dots, v_{\ell-1}\rangle. \tag{2.2.10}$$

Let $\rho_\ell \colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$ be the Weil representation:

$$\begin{aligned} \rho_\ell(T)v_j &= e^{\pi i v^2} e_j \\ \rho_\ell(S)v_j &= \frac{\sqrt{i}}{\sqrt{\ell}} \sum_{k=0}^{\ell-1} e^{-2\pi i v_j\cdot v_k} e_k. \end{aligned}$$

Because V has an obvious basis, we can identify V with its dual space. With this identification, the dual of the representation ρ_ℓ will be denoted by $\rho_\ell^* \colon \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(V)$.

Definition 2.1. A modular form of weight $k \in \frac{1}{2}\mathbb{Z}$ and type ρ_ℓ is a holomorphic function $f\colon \mathbb{H} \to V$ such that

$$f(A\tau) = \varepsilon_A^{2k}(\tau)\rho_\ell(A, \varepsilon_A)f(\tau).$$

Since the v_j is an eigenbasis of V with respect to T, we can write

$$f(\tau) = \sum_j \sum_{n \in \frac{1}{N}\mathbb{Z}} a_{n,j} q^n v_j \tag{2.2.11}$$

where $\rho_\ell(T^N) = \mathrm{id}_V$. If $a_{n,j} = 0$ for all $n < 0$ and $j = 0, \dots, \ell - 1$, then we say f is holomorphic at infinity.

Definition 2.2. We denote by $\mathrm{Mod}(\mathrm{Mp}(\mathbb{Z}), k, \rho_\ell)$ the space of modular forms whose Fourier coefficients have finitely terms with negative exponent. Its subspace of holomorphic modular forms are denoted by $\mathrm{HolMod}(\mathrm{Mp}(\mathbb{Z}), k, \rho_\ell)$.

Remark 2.3. The space $\mathrm{HolMod}(\mathrm{Mp}(\mathbb{Z}), k, \rho_\ell)$ is finite-dimensional because it can be identified with the global sections of a vector bundle on a complete modular curve.

2.3. **Noether–Lefschetz numbers are modular.** Fix ℓ and a any proper curve $C \to \mathcal{F}_\ell$. Recall $\Delta_\ell(h, d) = d^2 - \ell(2h - 2)$. We break the Noether–Lefschetz numbers into ℓ groups and define

$$b_{n,j} = \begin{cases} C \cdot D_{h,d} & \text{if } n = \frac{1}{2\ell}\Delta_\ell(h,d) \text{ and } d \equiv j \pmod{\ell}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.12}$$

Now, for each $j = 1, \dots, \ell - 1$, we define the following generating series

$$\Phi_j(q) = \sum_{n \in \frac{1}{2\ell}\mathbb{N}} b_{n,j} q^n. \tag{2.3.13}$$

Theorem 2.4 (Borcherds)**.** *There is a modular form of weight* $21/2$ *and type* ρ_ℓ^* *whose Fourier expansion at* $i\infty$ *is*

$$\Phi(q) = \sum_{r=1}^{\ell} \Phi_r(q) v_r \in \mathbb{C}[[q^{1/2\ell}]] \otimes \mathbb{C}[\Lambda^\vee/\Lambda]. \tag{2.3.14}$$

The advantage of this point of view is that the space of vector-valued modular forms, as it appears here, is much smaller than the scalar counterpart. Compare the following lemma to the 22 dimensional problem we encountered earlier for $\ell = 4$.

Lemma 2.5 (Maulik–Pandharipande)**.** *For* $\ell = 2, 4, 6, 8$ *the dimension of the space* $\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}), 21/2, \rho_\ell^*)$ *is* $2, 3, 4, 5$ *respectively. Moreover, we can give a simple and explicit basis for these spaces.*

Maulik and Pandharipande give an explicit basis in terms of Rankin–Cohen brackets of Siegel theta functions and Eisenstein series. In other words, provided we can compute a few of the initial coefficients of $\Phi(q)$, we can compute all coefficients readily.

We need so few coefficients now that the Gromov–Witten approach is overkill. Classical geometry will be sufficient. The following three sources provide enough initial conditions for all the four cases ($\ell = 2, 4, 6, 8$) above:

- The coefficient of $q^0 v_0$ is the degree of the Hodge bundle.
- The coefficient of $q^1 v_1$, corresponding to $C \cdot \mathrm{NL}_{0,0}$, is the number of "singular" fibers (i.e., polarization degenerates).
- The Castelnuovo bound restricts the genus of a space curve by its degree. In particular, if $h > \binom{d-1}{2}$ then $C \cdot \mathrm{NL}_{h,d} = 0$. This gives a lot of zero coefficients for free.

3. Borcherds' theorem in general

We used Theorem 2.4 to deduce modularity of intersection numbers with Noether–Lefschetz loci in quasi-polarized K3 surfaces. This is a special instance of Theorem 3.3. The latter applies to intersection numbers of special divisors, generalizing the Noether–Lefschetz divisors, in the moduli of hyperkähler varieties.

3.1. **Heegner divisors.** Let Λ be an even lattice of signature $(2, s)$, $s > 0$. The period domain is the semi-algebraic set

$$\mathcal{D}_\Lambda \stackrel{\text{def}}{=} \{[w] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid w^2 = 0,\ w \cdot \overline{w} = 0\}. \tag{3.1.15}$$

The dual of the lattice Λ is denoted by $\Lambda^\vee \subset \Lambda \otimes \mathbb{Q}$. We fix a discrete group $\Gamma \subset O(\Lambda)$ fixing the discriminant $D(\Lambda) \stackrel{\text{def}}{=} \Lambda^\vee/\Lambda$, where $O(\Lambda)$ is the orthogonal group of Λ. We will be interested in special divisors on the quotient $Y_\Gamma \stackrel{\text{def}}{=} \Gamma \backslash \mathcal{D}_\Lambda$.

Each $x \in \Lambda^\vee$ defines a hyperplane $x^\perp \subset \mathbb{P}(\Lambda \otimes \mathbb{C})$ and consequently a hyperplane section

$$\mathcal{D}_x \stackrel{\text{def}}{=} \mathcal{D} \cap x^\perp = \{[w] \in \mathcal{D} \mid w \cdot x = 0\}. \tag{3.1.16}$$

The orbit of the $\mathcal{D}_x$ leads us to consider the following *Heegner divisor* on Y_Γ:

$$y_{n,\gamma} \stackrel{\text{def}}{=} \Gamma \backslash \left(\bigcup_{\substack{x \in \gamma \\ x^2 = 2n}} \mathcal{D}_x \right), \tag{3.1.17}$$

where $\gamma \in D(\Lambda)$ and $n \in \mathbb{Q}$. If $\mathcal{D}_x$ is non-empty then $x^2 < 0$, so we set $y_{0,\gamma} = 0$ except for $y_{0,0}$ which is defined to be $\Gamma \backslash\, \mathcal{O}(1)$.

Remark 3.1. When considering moduli of K3 surfaces, our Λ had signature $(2, 19)$. The Heegner divisors $y_{n,\gamma}$ correspond to the divisors $D_{h,d}$, where $\gamma = d \mod \ell$ and $n = -\frac{\Delta_\ell(h,d)}{2\ell}$.

The Borcherds' theorem we are alluding to can be summarily stated as below. We will make it precise and outline its proof in this section.

Theorem 3.2. *The Heegner divisors are the Fourier coefficients of a vector-valued modular form.*

3.2. **The Weil representation.** The group ring $V \stackrel{\mathrm{def}}{=} \mathbb{C}[D(\Lambda)]$ of the discriminant group $D(\Lambda)$ admits a standard representation

$$\rho_\Lambda\colon \mathrm{Mp}_2(\mathbb{Z}) \to V \tag{3.2.18}$$

called the Weil representation. Using the notation from Section 2.2 it is defined by:

$$\begin{aligned} \rho_\Lambda(T)v_\gamma &= e^{\pi i v^2} e_\gamma \\ \rho_\Lambda(S)v_\gamma &= \frac{\sqrt{i}^{s-2}}{\sqrt{\#D(\Lambda)}} \sum_{\delta\in D(\Lambda)} e^{-2\pi i\gamma\cdot\delta} e_\delta. \end{aligned}$$

As before, we identify V with its dual and denote the dual representation by ρ_Λ^*. From now on ρ will denote either ρ_Λ or ρ_Λ^*. The only cusp for these representations is the one at $i\infty$.

Depending on Λ, there is a minimal positive integer N such that

$$\rho(T^N) = 1. \tag{3.2.19}$$

Then, any modular form $f \in \mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}), k, \rho_\Lambda)$ admits a Fourier series expansion

$$f(\tau) = \sum_{\substack{n\in\frac{1}{N}\mathbb{Z}\\ \gamma\in D(\Lambda)}} a_{n,\gamma} q^n e_\gamma \tag{3.2.20}$$

where $V = \mathbb{C}\langle e_\gamma \mid \gamma \in D(\Lambda)\rangle$, $\tau \in \mathbb{H}$ and $q = \exp(2\pi i\tau)$. Here $q^{1/N}$ should be viewed as atomic, being the uniformizer of an appropriate quotient of $\mathbb{H}$ at the image of $i\infty$.

We will write $\mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}), k, \rho)$ for the modular forms on $\mathbb{H}$ that are meromorphic at the cusp. The space of modular forms holomorphic at the cusp are denoted $\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}), k, \rho)$.

3.3. **Explicit form of Borcherds' theorem.** Let us denote by λ the following function on holomorphic modular forms that computes the Fourier coefficients at $i\infty$.

$$\lambda\colon \mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}), k, \rho_\Lambda) \to \mathbb{C}[[q^{1/N}]]. \tag{3.3.21}$$

We can now state Borcherds' theorem precisely. Define the formal power series

$$\Phi(q) = \sum_{n\in\frac{1}{N}\mathbb{N},\,\gamma} y_{-n,\gamma} q^n e_\gamma \in \mathrm{Pic}(Y_\Gamma) \otimes \mathbb{C}[[q^{1/N}]] \otimes V, \tag{3.3.22}$$

where we identified each Heegner divisor with its divisor class in $\mathrm{Pic}(Y_\Gamma)$.

Theorem 3.3. *The series $\Phi(q)$ is contained in*

$$\mathrm{Pic}(Y_\Gamma) \otimes \lambda\left(\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}), 1 + \tfrac{s}{2}, \rho_\Lambda^*)\right). \tag{3.3.23}$$

4. Outline of the proof of Borcherds' theorem

4.1. **Principal Heegner divisors.** Let $\mathrm{He}(Y_\Gamma) = \mathbb{Z}\langle y_{n,\gamma} \mid n, \gamma\rangle$ denote the group freely generated by the Heegner divisors. We will write $\mathrm{PHe}(Y_\Gamma) \subset \mathrm{He}(Y_\Gamma)$ for the subgroup of *principal* Heegner divisors. Naturally, there is an injection

$$\mathrm{He}(Y_\Gamma)/\,\mathrm{PHe}(Y_\Gamma) \hookrightarrow \mathrm{Pic}(Y_\Gamma). \tag{4.1.24}$$

Remark 4.1. The image is denoted by $\mathrm{Pic}^{\mathrm{NL}}(Y_\Gamma)$ in [1], because the Heegner divisors are the Noether–Lefschetz divisors in the moduli of K3 surfaces.

Let us define a map that strips off positive terms from the Fourier expansion of a modular form

$$\lambda^- \colon \mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}), k, \rho_\Lambda) \to \mathbb{C}[[q^{-1/N}]] \otimes V. \tag{4.1.25}$$

We also have an obvious surjective map

$$\xi \colon \mathbb{C}[[q^{-1/N}]] \otimes V \twoheadrightarrow \mathrm{He}(Y_\Gamma) : q^n e_\gamma \mapsto y_{n,\gamma}. \tag{4.1.26}$$

The following remarkable theorem of Borcherds states that the Laurent tails of meromorphic modular forms of weight $1 - s/2$ and type ρ_Λ record linear relations between Heegner divisors modulo principal divisors.

Theorem 4.2 (Borcherds)**.**

$$\xi\lambda^- \left(\mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}), k, \rho_\Lambda)\right) \subset \mathrm{PHe}(Y_\Gamma). \tag{4.1.27}$$

Remark 4.3. The statement here is a simplified form of [3, Theorem 4.1] in light of McGraw's thesis work [7] which states that the space of meromorphic forms here have a basis having only integer coefficients appearing in their Laurent tails.

Remark 4.4. The proof of this theorem relies on Borcherds' "theta lifting" in [2], where he constructs explicit automorphic forms with poles and zeros consisting only of Heegner divisors.

With certain restrictions on the lattice and the group Γ, all principal divisors are obtained by Borcherds' theta lifting.

Theorem 4.5 (Bruinier [4])**.** *Suppose the lattice Λ contains two copies of the hyperbolic plane with an even definite orthogonal complement. Let Γ be the largest subgroup of $O(\Lambda)$ that fixes the discriminant $D(\Lambda)$. Then the containment in Theorem* (4.1.27) *is an equality.*

4.2. **Duality.** With ρ denoting either ρ_Λ or ρ_Λ^* we define the following spaces:

$$\mathrm{Pow}(\rho) = \mathbb{C}[[q^{1/N}]] \otimes V \tag{4.2.28}$$

$$\mathrm{Lau}(\rho) = \mathbb{C}[[q^{1/N}]][q^{-1/N}] \otimes V \tag{4.2.29}$$

$$\mathrm{Sing}(\rho) = \mathrm{Lau}(\rho)/q^{1/N}\,\mathrm{Pow}(\rho). \tag{4.2.30}$$

Recall the map λ from modular forms into Pow that expands out the Fourier coefficients and the map λ^- that extracts the Laurent tail of meromorphic modular forms to $\mathrm{Sing}(\rho)$.

Define the obstruction space to realizing Laurent tails as modular forms

$$\mathrm{Obst}(k,\rho) = \mathrm{Sing}(\rho)/\lambda^{-}\left(\mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}),k,\rho)\right). \tag{4.2.31}$$

There is a natural pairing

$$\mathrm{Pow}(\rho)\times\mathrm{Sing}(\rho)\to\mathbb{C} \tag{4.2.32}$$

$$(f,g)\mapsto \text{constant coefficient of } fg. \tag{4.2.33}$$

Here we let the V's pair canonically so $fg\in\mathbb{C}[[q^{1/N}]][q^{-1/N}]$. We read of the coefficient of q^0.

Theorem 4.6. *The space* $\mathrm{Obst}(2-k,\rho)$ *is dual to* $\lambda\,\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}),k,\rho^*)$. *In other words,* $\lambda^{-}\,\mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}),2-k,\rho)$ *and* $\lambda\,\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}),k,\rho^*)$ *are the annihilators of one another with respect to the natural pairing above.*

Sketch of proof. There is a proper curve C, the compactification of a quotient of the upper half plane, and a vector bundle $\mathcal{F}$ on C such that global sections of $\mathcal{F}$ are identified with $\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}),2-k,\rho)$. If D is a divisor on C supported on the cusp then the global sections of $\mathcal{F}(D)$ are some of the meromorphic modular forms with pole order bounded by the degree of D. The natural exact sequence

$$0\to\mathcal{F}\to\mathcal{F}(D)\to\mathcal{F}(D)|_D\to 0 \tag{4.2.34}$$

gives rise to the following exact sequence

$$0\to\mathrm{H}^0(\mathcal{F})\to\mathrm{H}^0(\mathcal{F}(D))\to\mathrm{H}^0(\mathcal{F}(D)|_D)\to\mathrm{H}^1(\mathcal{F})\to\mathrm{H}^1(\mathcal{F}(D))\to 0. \tag{4.2.35}$$

Observe that the term $\mathrm{H}^0(\mathcal{F}(D)|_D)$ is precisely the Laurent tails of meromorphic forms. Moreover, with $\deg D\gg 0$, the last term $\mathrm{H}^1(\mathcal{F}(D))$ vanishes as D becomes very ample. Therefore, $\mathrm{H}^1(\mathcal{F})$ is identified with the obstruction space $\mathrm{Obst}(2-k,\rho)$.

Serre duality gives $\mathrm{H}^1(\mathcal{F})^\vee = \mathrm{H}^0(\omega_C\otimes\mathcal{F}^\vee)$. Since ω_C has weight 2, the last space is $\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}),k,\rho^*)$. ☐

Remark 4.7. This result implies that the dimension of the group of Heegner classes $\mathrm{He}(Y_\Gamma)/\,\mathrm{PHe}(Y_\Gamma)$ is finite. In fact, if the conditions of Theorem 4.5 hold, then its dimension is easily computed from the dimension of a space of holomorphic modular forms.

4.3. **Proving Theorem 3.3.** Take $g\in\mathrm{Mod}(\mathrm{Mp}_2(\mathbb{Z}),1-s/2,\rho_\Lambda)$, so that in light of Theorem 4.2 its Laurent tail $\lambda^{-}(g)$ records the coefficients of a (finite) linear combination of Heegner divisors that is principal. That is, if

$$\lambda^{-}(g)=\sum_{n<0,\gamma} a_{n,\gamma}q^n e_\gamma, \tag{4.3.36}$$

then

$$\sum_{n<0,\gamma} a_{n,\gamma}y_{n,\gamma}\equiv 0. \tag{4.3.37}$$

That means $\lambda^-(g)$ annihilates $\Phi(q)$ with respect to the natural pairing (4.2.32). This too can be spelled out:

$$\Phi(q) \cdot \lambda^-(g) = \sum_{n<0,\gamma} \big(y_{n,\gamma} q^{-n} e_\gamma\big) \cdot (a_{n,\gamma} q^n e_\gamma) \tag{4.3.38}$$

$$= \sum_{n<0,\gamma} a_{n,\gamma} y_{n,\gamma} \equiv 0. \tag{4.3.39}$$

It now follows from Theorem 4.6 that $\Phi(q) \in \mathrm{Pic}(Y_\Gamma) \otimes \lambda\,\mathrm{HolMod}(\mathrm{Mp}_2(\mathbb{Z}), 1 + s/2, \rho_\Lambda^*)$.

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