MPI MiS mini-course: Hodge theory and periods of varieties
Exercise set 4

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Numbered theorems and exercises are with reference to [1].

1. (a) Use the Mayer-Vietoris[1] sequence to compute the cohomology of S^1 (the unit circle).

   (b) Let E be an elliptic curve, and let U, V be simply connected open subsets of E such that $U \cup V = E$ and $U \cap V$ consists of two cylindrical bands. Write down the Mayer-Vietoris sequence to compute the cohomology of E.

   (c) Let $X, Y \subseteq \mathbb{A}^n_{\mathbb{C}}$ be distinct smooth irreducible affine varieties. Show that there exists an open neighborhood $X \cap Y \subseteq U$ of $X \cup Y$ such that $X \cap Y$ is a deformation retract of U. Explain why the Meyer-Vietoris sequence on [1, page 63] is valid. Is there always an open neighbourhood of $X \cap Y$ that admits a deformation retract onto $X \cap Y$ if X, Y are path connected topological spaces?

2. Let $X\colon 0 = T^3 - f(x, y, z, w) \subseteq \mathbb{P}^4$ be a μ_3-branched cover of a smooth cubic surface X. Let $\sigma\colon X \to X$ be a generator for the automorphism group over the branch locus $S\colon 0 = f(x, y, z, w) \subseteq \mathbb{P}^3$. Show that σ^* acts on the cohomology of X. Show that the only σ^*-invariant classes in $\mathrm{H}^3(X, \mathbb{R})$ are pullbacks of classes from $\mathbb{P}^3$.

3. Exercise 1.4.2.

   Consider an algebraic surface X with one isolated singularity at a point p. Let $\widetilde{X}$ be a resolution of singularities, and denote the inverse image of p in the resolution by E. The inverse image, the so-called exceptional locus, is an algebraic curve. Let us assume that it is smooth. Discuss the mixed Hodge structure on X as an extension of a weight 1 Hodge structure by one of weight 2. Show that we get an ordinary Hodge structure if the singularity is an ordinary double point, but a genuine mixed Hodge structure if the singularity is an ordinary m-fold point with $m > 2$.

   Warning: There is at least one typo in the section of the book which is helpful for this problem. Also, I keep getting an extension of a weight 2 mixed hodge structure by a weight 1 mixed hodge structure when I try to solve this. That said, I think we're up for a challenge on Friday!

References

[1] James Carlson, Stefan Müller-Stach, and Chris Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, vol. 168, Cambridge University Press, Cambridge, 2017. Second edition of [MR2012297]. MR3727160

[1] Fun fact: Leopold Vietoris lived to be 110 years old, and held the record for being the oldest confirmed living Austrian.