MPI MiS mini-course: Hodge theory and periods of varieties
Exercise set 2

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Numbered theorems and exercises are with reference to [1].

Ex. A Complete the proof of Theorem 1.2.1.

1. Let C_t be the family of hyperelliptic curves over $\mathbb{A}^1_{\mathbb{C}} := \operatorname{Spec} \mathbb{C}$ defined by

$$C_t \colon y^2 = x(x-t)(x^3 + 10^5).$$

   (a) Compute the variety $\Delta \subseteq \mathbb{A}^1_{\mathbb{C}}$ of t such that C_t is singular. What is the genus of C_t, for $t \notin \Delta$?

   (b) Use the `sage`[1] function `C.period_matrix()` to compute the period matrix of C_t, for a given $t \notin \Delta$. Examples and source code available here:

   `https://github.com/nbruin/examplesNumericalEndomorphisms`.

   (c) Compute the genus of the normalization of C_0. Compute the Riemann matrix.

   (d) Experiment with computing Riemann matrices as $t \to 0$.

   (e) Can you find a matrix in the integral symplectic group $\operatorname{Sp}(4, \mathbb{Z})$ to confirm your suspicions?

2. Let $\mathcal{E}_t$ be the family of elliptic curves defined by

$$\mathcal{E}_t \colon y^2 = w^3 + (t^2 + t + 1)w^2 + (t^6 + t).$$

   Show that the total space is smooth. Show that the fibre $\mathcal{E}_0$ is singular. Compute the monodromy operator around $t = 0$.

3. Exercise 1.2.7.

   Show that if the module A has a Hodge structure of weight n and the module B has one of weight m, the module $\operatorname{Hom}(A, B)$ inherits a Hodge structure of weight $m - n$ by setting

$$\operatorname{Hom}(A, B)^{i,j} := \{f \colon A \to B : f(A^{p,q}) \subseteq B^{p+i, q+j}\}.$$

   Show that this is in agreement with our definition for the Hodge structure on the dual of A, thereby showing that this indeed defines a Hodge structure of the asserted weight.

References

[1] James Carlson, Stefan Müller-Stach, and Chris Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, vol. 168, Cambridge University Press, Cambridge, 2017. Second edition of [MR2012297]. MR3727160

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[1] Version 8.4 or later