MPI MiS mini-course: Hodge theory and periods of varieties
Exercise set 1

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Numbered theorems and exercises are with reference to [1].

Ex. A Finish the technical details of the proof from lecture. i.e, For the ω_λ, ρ, U from the "Derivative of the period map section", show

$$\int_U d\left(\frac{\rho\omega}{y}\right) = -\int_{|y|=\frac{\epsilon}{4}} \frac{\rho\omega}{y}.$$

1. Exercise 1.1.7.

   A holomorphic one-form on a Riemann surface is a differential one-form which is locally given by $f(z)dz$, where z is a local holomorphic coordinate. Show that such a form is closed. Formulate and investigate the analogous assertion(s) for holomorphic forms on a complex manifold of complex dimension 2.

2. Exercise 1.1.10.

   Let S be a Riemann surface, and let $A \subseteq S$ be a nonempty finite set. Show that there is an exact sequence res

   $$0 \longrightarrow \mathrm{H}^1(S) \longrightarrow \mathrm{H}^1(S - A) \longrightarrow \mathrm{H}^0(A) \longrightarrow \mathrm{H}^0(S) \longrightarrow 0$$

   Note that elements of $H^0(A)$ are linear functionals on the vector space spanned by the points of A. They can be viewed as the pointwise residue as defined previously, and they can be combined to form the globally defined map res. Your argument should show that the above sequence is defined on the level of integral cohomology.

3. Exercise 1.1.11.

   Consider the degeneration of elliptic curves $\mathcal{E}_t$ defined by $y^2 = x^3 - t$. Find all values of t for which $\mathcal{E}_t$ is singular. By drawing a series of pictures of branch cuts, show that the monodromy transformation for $t = 0$ has order six, and find the corresponding matrix.

4. Exercise 1.1.12.

   Let $\{\mathcal{E}_t\}$ be a family of elliptic curves with just two singular fibers, one at $t = 0$, the other at $t = \infty$. Show that the complex structure of $\mathcal{E}_t$ does not vary.

References

[1] James Carlson, Stefan Müller-Stach, and Chris Peters, *Period mappings and period domains*, Cambridge Studies in Advanced Mathematics, vol. 168, Cambridge University Press, Cambridge, 2017. Second edition of [MR2012297]. MR3727160