MPI MiS mini-course: Hodge theory and periods of varieties
Exercise set 0

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1.1 Miscellaneous

1. Let $E\colon y^2 = x^3 + ax + b$ be a non-singular (affine) complex elliptic curve. Show that $\omega := \frac{dx}{y}$ is a holomorphic differential on E.

2. Let $f(x)$ be a monic square-free polynomial of degree $d = 2g+2$, where $g \geq 1$. Let $C_0\colon y^2 = f(x)$ be a hyperelliptic curve in affine space. Consider also the affine plane curve defined by $C_1\colon t^2 = s^d f(1/s)$.

   Prove that the maps
   $$\phi_{01}\colon (x,y) \mapsto \left(\frac{1}{x}, \frac{x}{y^{g+1}}\right)$$
   $$\phi_{10}\colon (s,t) \mapsto \left(\frac{1}{s}, \frac{t}{s^{g+1}}\right)$$
   satisfy the gluing axioms, and that C_0, C_1 glue together into a projective curve. (You may assume that the resulting scheme is separated, as it is clearly Hausdorff with respect to the euclidean topology). Show this projective curve is smooth.

3. Show that the exact same models and gluing maps give a smooth curve when $d = 2g+1$.

4. Assume for simplicity that d is even. Extend the differential $(C_0, x^i \frac{dx}{y})$ to a meromorphic differential on all of C. Show that it is holomorphic if and only if $0 \leq i < \frac{d-1}{2}$.

1.2 Stokes' theorem and applications

1. Remember what Stokes' theorem is[1].

2. Use Stokes' theorem to "prove"[2] the fundamental theorem of calculus.

3. Prove the residue theorem:

   Theorem 1.1. *Let X be a smooth compact Riemann surface and ω a non-zero meromorphic differential. Prove that*
   $$\sum_{p \in X} \operatorname{res}_p \omega = 0.$$

   As a corollary, conclude that $\#\{\text{zeros of } f\} - \#\{\text{poles of } f\} = 0$, counted with multiplicity.

[1] This might help: https://www.smbc-comics.com/comic/2014-02-24

[2] For the people concerned about the logic here, you may add Stokes' theorem to your ZFC axiom set.

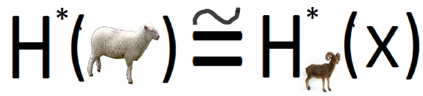


Figure 1: In the future, we find out that cohomology with coefficients in the constant sheep is de Ram cohomology.

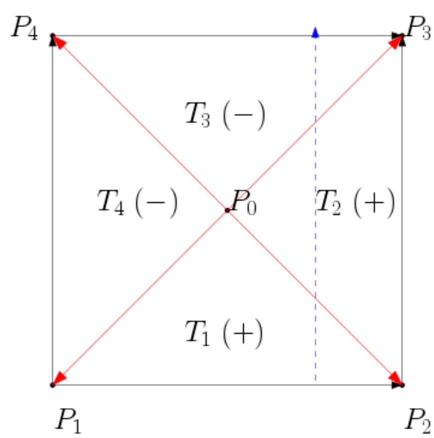
1.3 The de Rham theorem

1. Let $f(z) := f(x+iy) = u(x,y) + iv(x,y)$ be a real-differentiable function. Define the *Wirtinger derivatives*
$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar z} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$
Show that f is holomorphic if and only if $\frac{\partial f}{\partial \bar z} = 0$.

2. Remember the de Rham theorem. Find a non-trivial differential 1-form ω dual to the homology class of a simple loop around 0 in $\mathbb{C}\backslash\{0\}$. That is, dual in a sense of a linear functional
$$\omega\colon \gamma \mapsto \int_\gamma \omega$$
Show $\bar\omega$ is a multiple of ω as a functional on loops. This is not our typical story in this course! (Moral: beware of punctures and singularities.)

3. Let X be a smooth compact Riemann surface. Convince yourself that $H^n(X,\mathbb{Z})$ is canonically isomorphic to $\mathrm{Hom}(H_n(X,\mathbb{Z}),\mathbb{Z})$ via the universal coefficient theorem. Find a smooth real manifold X for which this is false.

4. Let X be the smooth compact Riemann surface of genus 1. Depicted below is an oriented "triangularization" of X:



As $P_1 = P_2$ and $P_3 = P_4$, these are not really triangles, but we will still call them triangles since they are the images of oriented simplicies. The symbol next to each triangle $(+/-)$ describes its orientation with respect to the oriented edges. For instance, with $T_1 = [P_0, P_1, P_2]$, we have that
$$\begin{aligned}\partial T_1 &= [P_0,P_1] - [P_0,P_2] + [P_1,P_2]\\ &= [P_0,P_1] + [P_2,P_0] + [P_1,P_2]\end{aligned}$$
while
$$\partial T_4 = -\left([P_0,P_1] - [P_0,P_4] + [P_1,P_4]\right).$$

Define $f \in \mathrm{Hom}(C_1(X), \mathbb{Z})$ by extending

$$f(e) = \begin{cases} 1 & \text{if } e = [P_1, P_2] \\ 1 & \text{if } e = [P_0, P_2] \\ 1 & \text{if } e = [P_0, P_3] \\ 1 & \text{if } e = [P_4, P_3] \\ 0 & \text{otherwise} \end{cases}$$

with $\mathbb{Z}$-linear combinations. Note that $[P_4, P_3] = [P_1, P_2]$. Visually, we can think of f as taking a value 1 whenever it meets the blue vertical cycle.

(a) Explain why the labelling $\Delta = [p, q, r]$ is the unique description such that $[p, q]$, $[q, r]$ have the same direction and $[p, r]$ has the reverse direction.

(b) Show that f defines a 1-cocycle.

(c)

Definition 1.2. Let $f, g \in C^1(X, \mathbb{Z})$ be 1-cocycles. Define on oriented simplicies

$$(f \cup g)([p, q, r]) := f([p, q]) \cdot g([q, r])$$

and extend $\mathbb{Z}$-linearly to define the *cup product*.

Show that $(f \cup f)(X) = 0$.

(d)

Definition 1.3. Let $f \in C^r(X, \mathbb{Z})$ be an r-cochain and let $\sigma := [p_0, \ldots, p_{r+s}] \in C_{r+s}(X, \mathbb{Z})$ be an oriented $(r+s)$-simplex. Define

$$\sigma \cap f := f([p_0, \ldots, p_r]) \cdot [p_r, \ldots, p_{r+s}]$$

and extend $\mathbb{Z}$-linearly to define the *cap product*.

Let $[X] \in C_2(X, \mathbb{Z})$ be the 2-chain class of the surface X. Compute $([X] \cap f) \cap f$ and $[P_1, P_2] \cap f$. Also compute the cycle intersections $([X] \cap f) \cap ([X] \cap f)$ and $[P_1, P_2] \cap ([X] \cap f)$.

(e) Remember what Poincaré duality is. Use Poincaré duality to explain your observations in part (d).

Further notes: It turns out that $f \cup g = g \cup f = 0$ whenever g is a coboundary, and thus the cup product descends to cohomology. Similarly, the cap product id also defined on homology and cohomology. If X is a smooth compact Riemann surface, then we have two natural ways to view the cup product. First, the de Rham isomorphism gives $[\omega_1 \wedge \omega_2] = [\omega_1] \cup [\omega_2]$. Second, Poincaré duality gives $(\gamma \cap \delta)^* = \gamma^* \cup \delta^*$, where $\cap$ is the cycle intersection.