BLOCK COURSE ON
HODGE THEORY AND PERIODS OF VARIETIES

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1. Program

This is a block course that will take place twice during the spring semester 2019 at MPI MiS Leipzig. Each block will consist of a single workweek with 4 hours of lectures every day, Monday through Friday. In the afternoons there will be a 2 hour exercise session. The first few lectures can be enjoyed with some knowledge of calculus and rudimentary visualization skills.

2. Overview

Starting from 19th century, elliptic and hyperelliptic integrals captivated the imagination of almost every mathematician. The initial observations provided the intuition for much of the development of modern complex geometry. At our present day, one starts learning the subject from these modern abstractions. However, in this lecture, we will work with concrete examples to see for ourselves what the ancients have seen in order to develop our intuition. This is intended so that we can anticipate what the abstract technical framework must look like, without doing anything technically demanding. We will end the first block by looking ahead and studying projective hypersurfaces, their Hodge decomposition and their periods.

The second block is technically independent from the first, but the first block serves to make the second block much more enjoyable. We will start the second block by laying the ground work for the definition of Hodge decomposition of smooth projective varieties. We will then continue with a brief study of algebraic cycles and of the Hodge conjecture, one of the million-dollar Millennium prize problems. Algebraic cycles appear in even degree of the cohomology and for odd degrees we will mimic the situation with curves and construct a complex torus, the intermediate Jacobian. Finally we will study variation of Hodge structures in earnest, both concrete and abstract.

3. Prerequisites

The first eight lectures assume very little technical knowledge and could in principle be enjoyed with undergraduate level knowledge. An acquaintance with singular homology and de Rahm cohomology would make things smoother, but is not necessary. The lectures nine and ten will demand more mathematical maturity as we will not prove everything, but it should be rewarding.

The second block is essentially a crash course into complex geometry and cohomology theory. Although the only prerequisite is linear algebra, the lectures will be demanding. Attending the first block should alleviate the difficulty.

Date: March 10, 2019.

4. Reading material

We will closely follow Chapter 1 [CMP17] for the first block except for the last lecture which uses a section from Chapter 3. The second block will be based on Chapters 2 and 3 of the same book.

For a historical and intuitive narrative, in the spirit of the first block, we also recommend [Mov]. The books of Voisin [Voi07a; Voi07b], especially the first volume, complement the second block very well.

5. First block

5.1. **Monday, lecture 1: Elliptic curves.** Construction of elliptic curves, topologically and algebraically. A hands on Hodge decomposition of cohomology. Periods of elliptic curves and elliptic integrals. Holomorphicity of the period map.

5.2. **Monday, lecture 2: Degenerating elliptic curves.** Asymptotics of periods as an elliptic curve degenerates. Picard–Fuchs equations and periods as values of a hypergeometric function. Moving around a singularity and the resulting Picard–Lefschetz transformation. The monodromy action on periods and the fundamental domain.

5.3. **Tuesday, lecture 3: Higher genus curves.** Topology of higher genus curves and the Hodge decomposition. The period map of genus g curves—this is also known as the Abel–Jacobi map. A degeneration of a genus two curve.

5.4. **Tuesday, lecture 4: Preparing for higher dimensions.** Giving names to our observations in preparation for higher dimensional varieties. The formal definition of a Hodge decomposition on a vector space, both mixed and pure.

5.5. **Wednesday, lecture 5: Topology of double planes.** The topology of two-fold branched coverings of the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, i.e., of "double planes".

5.6. **Wednesday, lecture 6: Hodge decomposition for double planes.** The Hodge decomposition on the cohomology of double planes. Curves in double planes and their cohomology classes. The first brush with Hodge conjecture.

5.7. **Thursday, lecture 7: Hodge theory of reducible surfaces.** Hodge decomposition and periods of reducible surfaces. Explicit computations with surfaces in space with two components.

5.8. **Thursday, lecture 8: Cubic surfaces.** There are many non-isomorphic cubic surfaces but they have no variation of Hodge structure. Nevertheless, human ingenuity has no limits: we will define an injective period map for the cubic surfaces.

5.9. **Friday, lectures 9 and 10: Hypersurfaces.** The cohomology and Hodge decomposition of a hypersurface closely relates to that of the ambient space. We will make this relation explicit. Lefschetz hyperplane theorem, The Hodge diamond, Griffiths residues, computing Hodge numbers using the Jacobian ring, computing periods.

6. Second block

6.1. **Monday, lecture 1: de Rahm cohomology of differentiable manifolds.** Tangent bundles, Riemannian metric and the de Rahm cohomology. Hodge star operator and harmonic forms.

6.2. **Monday, lecture 2: Kähler manifolds.** Decomposition of the complex tangent spaces. Hermitian metrics. The Kähler property. Hodge decomposition.

6.3. **Tuesday, lecture 3: Lefschetz and Dolbeaux decompositions.** Hard Lefschetz theorem and the induced decomposition. An algebraic interpretation of the Hodge decomposition via the Dolbeaux cohomology.

6.4. **Tuesday, lecture 4: Check cohomology.** Sheaves. Computing cohomologies using the Check cohomology of sheaves.

6.5. **Wednesday, lecture 5: Spectral sequences.** Definition of spectral sequences. Basic properties and some examples.

6.6. **Wednesday, lecture 6: Hypercohomology.** Making the Hodge decomposition fully algebraic via the hypercohomology of the Čheck-de Rahm complex.

6.7. **Thursday, lecture 7: Hypersurfaces in general.** Computing the cohomology of hypersurfaces of a general ambient space.

6.8. **Thursday, lecture 8: Mixed Hodge structures via log-poles.** Interpretation of the results for hypersurfaces via mixed Hodge structures, using the pole order grading of logarithmic differentials.

6.9. **Friday, lecture 9: Hodge conjecture and intermediate Jacobians.** Definition of algebraic cycles. A brief overview of Hodge conjecture. Defining the intermediate Jacobian. The Abel–Jacobi map for higher dimensional varieties.

6.10. **Friday, lecture 10: Variation of Hodge structures.** The variation of Hodge structures as we've witnessed them, followed by the abstract definition. Guiding open problems of the field.

References

[CMP17] J. Carlson, S. Müller-Stach, and C. Peters, *Period mappings and period domains*, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017, vol. 168, pp. xiv+562.

[Mov] H. Movasati, *A course in Hodge theory, with emphasis in multiple integrals*, to appear. [Online]. Available: `w3.impa.br/~hossein/myarticles/hodgetheory.pdf`.

[Voi07a] C. Voisin, *Hodge theory and complex algebraic geometry. I*, English, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007, vol. 76, pp. x+322, Translated from the French by Leila Schneps, ISBN: 978-0-521-71801-1.

[Voi07b] ——, *Hodge theory and complex algebraic geometry. II*, English, ser. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2007, vol. 77, pp. x+351, Translated from the French by Leila Schneps, ISBN: 978-0-521-71802-8.

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