TWISTOR CONNECTIVITY OF THE COHOMOLOGICAL MODULI SPACES

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Abstract. These notes are an expanded version of my talk at the Intercity Geometry Seminar in Netherlands, Spring of 2016. The goal of the seminar was to study Buskin's paper [Bus].

These notes provide a detailed exposition of Section 4 of [Bus] where twistor connectivity of the cohomological moduli spaces $\mathcal{M}_\phi$ is proven.

A large chunk of the calculation surrounding Lemma 4.9 in [Bus] can be bypassed. If you are already familiar with Buskin's paper jump to Section 4.3 to see this shortcut.

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Date: 09.04.2016.

Please email me at `emresertoz@gmail.com` if you have any suggestions or corrections.

1. Introduction

In this first section we will give a highly informal summary of Buskin's proof and an overview of the contents of these notes. Starting from Section 2 our arguments are fully rigorous.

1.1. **Summary of Buskin's proof.** The summary here is intended to put the content of these notes in perspective and to be short. I sacrificed historical and mathematical accuracy for the sake of readability, please use the original paper if you want either of those.

Let S and T be K3 surfaces and let $\psi : \mathrm{H}^2(S, \mathbb{Q}) \to \mathrm{H}^2(T, \mathbb{Q})$ be a rational Hodge isometry. Using Künneth's formula, the isometry ψ gives us a class $[\psi] \in \mathrm{H}^{2,2}(S \times T, \mathbb{Q})$. Buskin proved that the class $[\psi]$ is of analytic type (or algebraic when S and T are) as predicted by Hodge conjecture.

Buskin's proof can be summarized in 4 steps. The first step is to realize that the composition of Hodge isometries of analytic type are still of analytic type. As such, we find atomic pieces (called Hodge isometries of n-cyclic type) and proceed to show that all of these atomic pieces are of analytic type. The Hodge isometries that are atomic and of the same nature form a moduli space $\mathcal{M}_\phi$ (see Remark 3.2).

The second step is to construct, for each of the moduli spaces $\mathcal{M}_\phi$, one example of an atomic Hodge isometry that is of analytic type. The original construction is due to Mukai.

The third step is to show that each of these examples, when suitably deformed, remain of analytic type. The right notion of deformation here is via twistor lines, see Section 4.3 for more on this notion.

The last step is to prove that the moduli spaces $\mathcal{M}_\phi$ are connected via twistor lines (see Theorem 5.10). Hence, by the previous step all the atomic pieces are of analytic type. Then by the first step every Hodge isometry is of analytic type.

1.2. **Content of these notes.** In these notes we focus on step 4 of the summary above, i.e., we prove that the moduli spaces $\mathcal{M}_\phi$ (see Definition 3.1) are twistor connected. A crucial step is to find a way to deform a given Hodge isomorphism, which is done in Section 4.3.

From Section 2 until Section 4.3 we cover the basic material to define the relevant objects. From that point on, we specialize to the constructions involving pairs of K3s, linked by a "common" Kähler class. The Kähler class gives a sphere of complex structure on each of the K3s. An identification between these spheres allows one to deform the two K3s simultaneously. However, only one of these identifications allows a given Hodge isometry to preserve the Hodge decomposition on the cohomologies as the complex structures vary. This is proven in Theorem 4.25. The corresponding simultaneous deformation carves a line in the cohomological moduli space, which is called a twistor line.

Finally in Section 5 we tinker with established results on the twistor connectivity of the moduli space of K3 surfaces to construct twistor paths between any two points in $\mathcal{M}_\varphi$. This concludes the proof that $\mathcal{M}_\phi$ is twistor connected.

Since these notes are isolated from the rest of the proof of Buskin, we took the liberty in assuming that ϕ is any real isometry rather than a rational isometry of n-cyclic type. The proofs here work with this weaker hypothesis.

1.3. **Acknowledgments.** I would like to thank Daniel Huybrechts for his comments during and after the talk, the last section of these notes are significantly better as a result. I also would like to thank Bas Edixhoven for making me realize that the emphasis on Section 4.3 has to be on the *uniqueness* of the correct simultaneous deformation. Finally, I would like to thank Lenny Taelman who brought up a critical point that is now addressed in Remark 4.27.

2. Preliminaries: Moduli of K3s

Fix Λ, a representative of the unique isomorphism class of even unimodular lattices with signature $(3, 19)$. The inner product on Λ will be denoted by $q(\cdot,\cdot)$. As is usual, for any field k we will denote $\Lambda \otimes_{\mathbb{Z}} k$ by Λ_k.

A morphism of lattices $\eta : \Lambda' \to \Lambda$ extends to a k-linear morphism which we denote by $\eta_k : \Lambda'_k \to \Lambda_k$. Often we will drop the subscript of η_k as the context makes it clear.

Definition 2.1. A pair (S, η) is a *marked K3 surface* if S is a K3 surface and $\eta : \mathrm{H}^2(S, \mathbb{Z}) \to \Lambda$ is an isometry (the former lattice is taken with the topological intersection product). Two marked K3 surfaces (S, η) and (S', η') are called equivalent if there exists an isomorphism $f : S \to S'$ such that $\eta' = \eta \circ f^*$.

Notation 2.2. By $\mathcal{M}$ we denote the moduli space of (the equivalence classes of) marked K3s. It is a non-Hausdorff complex analytic space and a fine moduli space for the corresponding functor, see Section 7.2.1 of [Huy].

Definition 2.3. The set $C_\Lambda = \{l \in \Lambda_{\mathbb{R}} \mid q(l, l) > 0\} = C' \cup C''$ breaks into two disjoint cones such that $-C' = C''$. Distinguishing one of these components will be called an *orientation* on C_Λ and the distinguished cone is called the *positive cone of* Λ.

Definition 2.4. With S a K3 surface let $C_S = \{v \in \mathrm{H}^2(S, \mathbb{R}) \mid v \cdot v > 0\}$. Out of the two components of C_S only one of them contains the Kähler cone and as such we have a natural orientation on C_S with the component containing the Kähler cone chosen to be the *positive cone of* S, denoted C_S^+.

From this point onward, we assume that an orientation on C_Λ is fixed.

Definition 2.5. A marked K3 surface (S, η) is called *signed* if the isomorphism $\eta_{\mathbb{R}} : \mathrm{H}^2(S, \mathbb{R}) \to \Lambda_{\mathbb{R}}$ is orientation preserving between the cones C_S and C_Λ. Otherwise, (S, η) is called *unsigned*.

Remark 2.6. The moduli space $\mathcal{M}$ has two connected components, denoted $\mathcal{M}^+$ and $\mathcal{M}^-$, consisting of signed and unsigned marked K3s respectively. Notice that if $(S, \eta) \in \mathcal{M}^+$ then $(S, -\eta) \in \mathcal{M}^-$.

2.1. **The period map.**

Definition 2.7. The *period domain* of Λ is the set $\Omega_\Lambda = \{[l] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid q(l, l) = 0, q(l, \bar{l}) > 0\}$.

Definition 2.8. The natural holomorphic map $\pi : \mathcal{M} \to \Omega_\Lambda$ taking (S, η) to $[\eta(\mathrm{H}^{2,0}(S, \mathbb{C}))]$ is called the *period map.*

The following two major results will be evoked without further mention. Their proofs are in [Huy], labeled Theorem 7.4.1 and 7.5.3 respectively.

Theorem 2.9 (Surjectivity)**.** *The period map π is surjective.*

Theorem 2.10 (Global Torelli)**.** *If $\pi(S, \eta) = \pi(S', \eta')$ then $S \simeq S'$.*

3. Preliminaries: Moduli of pairs of K3s

Let $\phi : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}}$ be a real isometry preserving the orientation of C_Λ. For a K3 surface let $K_S \subset \mathrm{H}^2(S, \mathbb{R})$ denote the Kähler cone of S.

Definition 3.1. Let $\mathcal{M}_\phi \subset \mathcal{M} \times \mathcal{M}$ denote the moduli space of pairs $(S_1, \eta_1, S_2, \eta_2)$ such that the following hold:

- $\psi_\phi := \eta_2^{-1} \circ \phi \circ \eta_1 : \mathrm{H}^2(S_1, \mathbb{C}) \to \mathrm{H}^2(S_2, \mathbb{C})$ is a Hodge isometry.
- The image of the Kähler cone K_{S_1} intersects K_{S_2}, i.e., $\psi_\phi(K_{S_1}) \cap K_{S_2} \neq \emptyset$.

The space $\mathcal{M}_\phi$ is called the *cohomological moduli space.*

Remark 3.2. Perhaps the most natural way of viewing the cohomological moduli space $\mathcal{M}_\phi$ is to see it as the space of all Hodge isometries $\{\psi\}$ between marked K3s such that $\eta_2 \circ \psi \circ \eta_1^{-1} = \phi$.

Remark 3.3. Buskin takes ϕ to be a rational isometry (in fact of n-cyclic type), as this is what he needs. But to define the cohomological moduli space and to prove that it is twistor connected it suffices to assume that ϕ is a real isometry.

Remark 3.4. If a Hodge isometry $\psi_\phi : \mathrm{H}^2(S_1, \mathbb{R}) \to \mathrm{H}^2(S_2, \mathbb{R})$ sends a Kähler class to a Kähler class then ψ restricts to an orientation preserving map $C_{S_1} \to C_{S_2}$. Consequently, as ϕ is orientation preserving, if $(S_1, \eta_1, S_2, \eta_2) \in \mathcal{M}_\phi$ then either both η_i's are signed or both are unsigned.

Since $\mathcal{M}$ has two connected components $\mathcal{M}^-$ and $\mathcal{M}^+$ (see Remark 2.6), $\mathcal{M} \times \mathcal{M}$ has 4 which we will label as $\mathcal{M}^{++}, \mathcal{M}^{+-}, \mathcal{M}^{-+}, \mathcal{M}^{--}$ in the obvious manner. By the previous remark we conclude $\mathcal{M}_\phi \subset \mathcal{M}^{++} \coprod \mathcal{M}^{--}$.

Notation 3.5. Let $\mathcal{M}_\phi^+ = \mathcal{M}_\phi \cap \mathcal{M}^{++}$ and $\mathcal{M}_\phi^- = \mathcal{M}_\phi \cap \mathcal{M}^{--}$.

Our goal is to show that $\mathcal{M}_\phi^+$ and $\mathcal{M}_\phi^-$ are twistor connected. A notion that will be made precise in the coming sections. To facilitate the study of pairs of K3 surfaces, we will now define the space to keep track of their periods.

Definition 3.6. We will call the space $\Omega_\phi = \{([l], [\phi(l)]) \in \Omega_\Lambda \times \Omega_\Lambda\}$ the *twisted period domain.*

Remark 3.7. The map $\Omega_\Lambda \to \Omega_\Lambda$ taking $[l] \mapsto [\phi(l)]$ is well defined. To see this note that as ϕ is real, we have $\overline{\phi(x)} = \phi(\overline{x})$. Now use the fact that ϕ is an isometry. This shows that the projection maps $\Omega_\phi \to \Omega_\Lambda$ are surjective.

4. Twistor paths

Set-up 4.1. Fix a marked K3 surface (S, η) and let $\alpha \in \mathrm{H}^{1,1}(S, \mathbb{R})$ be a positive class, i.e., $\alpha \cdot \alpha > 0$. Denote by $\sigma_S \in \mathrm{H}^0(S, \Omega_S^2)$ a non-zero holomorphic differential form.

Notation 4.2. Define $V_{S,\alpha}$ to be the real vector space

$$\eta\left(\langle \alpha, \operatorname{Re}\sigma_S, \operatorname{im}\sigma_S \rangle\right) \subset \Lambda_\mathbb{R}.$$

Notation 4.3. Denote by $Q_{S,\alpha}$ the intersection $\mathbb{P}(V_{S,\alpha} \otimes \mathbb{C}) \cap \Omega_\Lambda$ within $\mathbb{P}(\Lambda_\mathbb{C})$. Since q restricts to a positive definite inner product on $V_{S,\alpha}$, the condition $q(l, \overline{l}) > 0$ is superfluous. On the other hand, $q(l, l) = 0$ cuts out a quadric curve on $\mathbb{P}(V_{S,\alpha} \otimes \mathbb{C}) \simeq \mathbb{P}^2$. Thus $Q_{S,\alpha} \simeq \mathbb{P}^1$.

Definition 4.4. The projective line $Q_{S,\alpha} \subset \Omega_\Lambda$ is called a *twistor line through* $\pi(S, \eta)$.

Remark 4.5. Observe that if $\pi(T, \mu) \in Q_{S,\alpha}$ then there exists a positive class $\beta \in \mathrm{H}^{1,1}(T, \mathbb{R})$ such that $Q_{S,\alpha} = Q_{T,\beta}$. To find β proceed as follows. By hypothesis any differential form $\sigma_T \in \mathrm{H}^{2,0}(T)$ will land in $V_{S,\alpha} \otimes \mathbb{C}$ and so the real and imaginary parts of σ_T will lie in $V_{S,\alpha}$. We may choose any $\mu(\beta)$ orthogonal to $\mu(\operatorname{Re}\sigma_T)$ and $\mu(\operatorname{im}\sigma_T)$ in $V_{S,\alpha}$.

Definition 4.6. If $q_0, q_1, \dots, q_n \in \Omega_\Lambda$ and for $i = 1, \dots, n$ each $Q_i \subset \Omega_\Lambda$ is a twistor line through q_{i-1} containing q_i, then the tuple $(Q_1, \dots, Q_n)$ is called a *twistor path* from q_0 to q_n.

If in addition the intermediate points q_i for $i = 1, \dots, n-1$ are the periods of generic K3s then the path is called a *generic twistor path.*

Despite the power of its implications, the proof of the following theorem is surprisingly simple. See Proposition 7.3.2 in [Huy].

Theorem 4.7. *Any two points of Ω_Λ may be connected by a generic twistor path.*

4.1. **Crash-course on hyperkähler structures.** Let X be a complex manifold and M the underlying real manifold. Then multiplication by $i \in \mathbb{C}$ induces an automorphism on the real tangent bundle TM which is denoted by $I : TM \to TM$. Note $I^2 = -\operatorname{Id}$.

One may recover X from the pair (M, I), and thus I is referred to as the *complex structure* of X (or on M). A Riemannian metric g on X is called I-invariant if $g(I(\cdot), I(\cdot)) = g$.

Definition 4.8. If g is I-invariant then $\omega = g(\cdot, I(\cdot))$ is a real $(1,1)$-form. If $\mathrm{d}\omega = 0$ then both the metric g and the form ω are called *Kähler*. In this case $[\omega] \in \mathrm{H}^{1,1}(X, \mathbb{R})$.

Definition 4.9. Suppose M admits 3 complex structures I, J and K which satisfy $IJ = -JI = K$, i.e., $\{\operatorname{Id}, I, J, K\}$ generate a quaternionic subalgebra of $\operatorname{End}(TM)$. The (ordered) triple (I, J, K) is called a *hypercomplex* structure on M.

Definition 4.10. Let M admit a hypercomplex structure (I, J, K) and suppose that there is a metric g on M invariant with respect to I, J and K. Then g is called a *hyperkähler metric* and the datum (g, I, J, K) is called a *hyperkähler structure* on M.

Remark 4.11. Given a hyperkähler structure (g, I, J, K) let $\mathbb{S} = \{aI + bJ + cK \mid a^2 + b^2 + c^2 = 1\}$ and notice that any $\lambda \in \mathbb{S}$ gives rise to a complex structure on M such that g is λ-invariant. As such, a hyperkähler structure defines a sphere of complex structures that invariate g.

4.2. **Kähler lines.** See Theorem 7.3.6 and the discussion after it in [Huy] for a more comprehensive treatment of what we will do here.

Recall Set-up 4.1. Let $S_\mathbb{R}$ denote the underlying real manifold of S and I the complex structure of S. If $\alpha \in \mathrm{H}^{1,1}(S, \mathbb{R})$ is a Kähler class then there exists a *unique* hyperkähler metric g on S with the associated Kähler form in α. Furthermore the associated sphere of complex structures $\mathbb{S}$ as described in Remark 4.11 depends only on g (although the choice of the hypercomplex structure (I, J, K) is not unique).

Notation 4.12. The sphere of complex structures, $\mathbb{S}$, on $S_\mathbb{R}$ is determined by α together with a complex structure. Moreover a sphere admits a unique complex structure. As such we will denote by $\mathbb{P}_{S,\alpha}$ the sphere $\mathbb{S}$ together with a complex structure.

For $\lambda \in \mathbb{S}$ let $S_\lambda = (S_\mathbb{R}, \lambda)$. Then there is a complex structure on the real manifolds $S_\mathbb{R} \times \mathbb{S} \to \mathbb{S}$ making this a family of K3 surfaces $\mathcal{S} \xrightarrow{\nu} \mathbb{P}_{S,\alpha}$ with the fiber over λ isomorphic to S_λ.

Let η be a marking of S. As the marking is defined at a topological level, while we vary the complex structure on $S_\mathbb{R}$ we may keep η fixed. Then the family ν can be marked with η, giving a family of marked K3 surfaces.

Notation 4.13. The resulting moduli map corresponding to the family ν of marked K3s will be denoted by $\Psi_{S,\alpha} : \mathbb{P}_{S,\alpha} \to \mathcal{M}$.

Definition 4.14. For $\lambda \in \mathbb{S}$ denote by ω_λ the Kähler form $g(\cdot, \lambda(\cdot))$. Let $\alpha_\lambda = [\omega_\lambda] \in \mathrm{H}^2(S_\mathbb{R}, \mathbb{R})$ be the associated Kähler class. As we can view $\alpha_\lambda \in \mathrm{H}^{1,1}(S_\lambda, \mathbb{R})$ we will later refer to α_λ as a *deformation of* α.

Remark 4.15. If $\mathbb{P}_{S,\alpha}$ passes through (T, μ) then there exists a Kähler class β on T such that $\mathbb{P}_{T,\beta} = \mathbb{P}_{S,\alpha}$. Indeed, we may assume $(T, \mu) = (S_\lambda, \eta)$. We will take $\beta = \alpha_\lambda$ so that we only have to show that the sphere of complex structures associated to α_λ is $\mathbb{S}$. This follows from the uniqueness of the hyperkähler metric: From α_λ we recover g and thus $\mathbb{S}$.

Lemma 4.16. *Let (I, J, K) be a hypercomplex structure arising out of a Kähler class α on S. Then $V_{S,\alpha} = \eta(\langle \alpha_I, \alpha_J, \alpha_K \rangle)$.*

Proof. As $\alpha_I = \alpha$ it suffices to prove that $\omega_J + i\omega_K \in \mathrm{H}^0(S, \Omega_S^2)$. This follows from the discussion before Proposition 13.3 in Chapter VIII of [Bar+04]. □

Remark 4.17. The previous reference allows one to calculate the norms of $\alpha_I, \alpha_J, \alpha_K$ and see that they are equal to one another. As such, for any $\lambda \in \mathbb{S}$ we have $\|\alpha_\lambda\| = \|\alpha\|$.

Lemma 4.16 implies that $V_{S,\alpha}$ does not depend on the complex structure on $S_\mathbb{R}$ nor on α, but on the hyperkähler metric g that is obtained from α. Since α_λ recovers g we have the following result.

Corollary 4.18. *For any $\lambda \in \mathbb{S}$ we have $V_{S,\alpha} = V_{S_\lambda,\alpha_\lambda}$. This implies $Q_{S,\alpha} = Q_{S_\lambda,\alpha_\lambda}$.*

Lemma 4.19. *$\pi \circ \Psi_{S,\alpha} : \mathbb{P}_{S,\alpha} \to Q_{S,\alpha}$ is an isomorphism.*

Proof. For any hypercomplex structure (I, J, K) on S, Lemma 4.16 makes this map explicit:

$$I, J, K \mapsto [\alpha_I], [\alpha_J], [\alpha_K],$$

which is clearly invertible. □

Definition 4.20. The twistor line $Q_{S,\alpha}$ is called a *Kähler twistor line* if α is a Kähler class.

Remark 4.21. In summary, whenever α is Kähler, the twistor line $Q_{S,\alpha} \subset \Omega_\Lambda$ admits a lift to $\mathcal{M}$. Namely, $\mathbb{P}_{S,\alpha}$.

4.3. **Simultaneous deformation of complex structures.** Fix an isometry $\phi : \Lambda_\mathbb{R} \to \Lambda_\mathbb{R}$. Take $(S, \eta, T, \mu) \in \mathcal{M}_\phi$ and let $\psi = \mu^{-1} \circ \phi \circ \eta$. Pick $\alpha \in K_S \cap \psi^{-1}(K_T)$, i.e., a Kähler class on S such that $\psi(\alpha)$ is also Kähler on T.

Notation 4.22. Given any isomorphism $\tau : \mathbb{P}_{S,\alpha} \to \mathbb{P}_{T,\psi(\alpha)}$ denote the graph of τ by $\Gamma_\tau \subset \mathbb{P}_{S,\alpha} \times \mathbb{P}_{T,\psi(\alpha)}$.

Using the natural maps $\mathbb{P}_{S,\alpha} \to \mathcal{M}$ and $\mathbb{P}_{T,\psi(\alpha)} \to \mathcal{M}$ we get a map $\Gamma_\tau \to \mathcal{M} \times \mathcal{M}$. As such we get to simultaneously deform (S, η) and (T, μ) over $\mathbb{P}^1 \simeq \Gamma_\tau$. Given $\lambda \in \Gamma_\tau$ denote the image of λ in $\mathcal{M} \times \mathcal{M}$ by $(S_\lambda, \eta, T_\lambda, \mu)$.

In fact, we are interested in deforming not just the K3 surfaces but also the Hodge isometry ψ. Noticing $\mathrm{H}^2(S, \mathbb{Q})$ and $\mathrm{H}^2(T, \mathbb{Q})$ are topological invariants we may define, for each $\lambda \in \Gamma_\tau$, the isometry $\psi_\lambda := \psi : \mathrm{H}^2(S_\lambda, \mathbb{Q}) \to \mathrm{H}^2(T_{\tau(\lambda)}, \mathbb{Q})$. However, as the complex structure changes so does the Hodge decomposition of the cohomology. Therefore, one may ask if ψ_λ is still a *Hodge* isometry. We will show that there is precisely one identification τ that makes ψ_λ a Hodge isometry for each λ.

Remark 4.23. Since $\psi(\mathrm{H}^0(S,\Omega_S^2)) = \mathrm{H}^0(T,\Omega_T^2)$ the map ϕ restricts to an isometry $V_{S,\alpha} \xrightarrow{\sim} V_{T,\psi(\alpha)}$, where we used Notation 4.2. In particular, ϕ restricts to an isomorphism $Q_{S,\alpha} \xrightarrow{\sim} Q_{T,\psi(\alpha)}$.

Definition 4.24. Let τ_α denote the composition of the series of isomorphisms

$$\mathbb{P}_{S,\alpha} \xrightarrow{\pi} Q_{S,\alpha} \xrightarrow{\phi} Q_{T,\psi(\alpha)} \xrightarrow{\pi^{-1}} \mathbb{P}_{T,\psi(\alpha)},$$

Where we used Lemma 4.19 to invert the last map. To comply with Buskin's notation we will denote the graph of τ_α by $\mathbb{P}_{\psi,\alpha} \subset \mathbb{P}_{S,\alpha} \times \mathbb{P}_{T,\psi(\alpha)}$.

Theorem 4.25. *The map $\Gamma_\tau \to \mathcal{M} \times \mathcal{M}$ yields a family of Hodge isometries $\{\psi_\lambda \mid \lambda \in \Gamma_\tau\}$ if and only if $\tau = \tau_\alpha$.*

Proof. Indeed ψ_λ is a Hodge isometry iff $\psi_\lambda(\mathrm{H}^{2,0}(S_\lambda,\mathbb{C})) = \mathrm{H}^{2,0}(T_\lambda,\mathbb{C})$. Letting $x_\lambda = \pi(S_\lambda,\eta)$ and $y_\lambda = \pi(T_\lambda,\mu)$ this last condition is equivalent to the statement that $\phi(x_\lambda) = y_\lambda$ since $\psi = \mu^{-1} \circ \phi \circ \eta$.

For $\lambda = (\lambda_1,\lambda_2) \in \Gamma_\tau$, identifying λ_i's with the marked K3 surfaces they represent, we conclude that ψ_λ is a Hodge isometry iff $\phi \circ \pi(\lambda_1) = \pi(\lambda_2)$. In other words, iff $\lambda_2 = \tau_\alpha(\lambda_1)$. □

Remark 4.26. This means that $\mathbb{P}_{\psi,\alpha}$ may be seen as a deformation of the Hodge isometry ψ.

Remark 4.27. As S and T deform so does the fourfold $S \times T$. Buskin defines sheaves on the deformation space of $S \times T$ to prove that Mukai's examples can be deformed. One may ask at this point, having bypassed a detailed analysis of the complex structures on $S \times T$ have we sacrificed a foothold in the later parts of Buskin's proof? In fact, we have not. All that will be needed later on in the paper is an understanding of how α and $\psi(\alpha)$ deform on S and T when we choose the simultaneous deformation corresponding to $\mathbb{P}_{\psi,\alpha}$.

With $\lambda = (\lambda_1,\lambda_2) \in \mathbb{P}_{\psi,\alpha}$, we will abuse Definition 4.14 and denote by $\alpha_\lambda := \alpha_{\lambda_1}$ and $\psi(\alpha)_\lambda := \psi(\alpha)_{\lambda_2}$ the deformations of the classes α and $\psi(\alpha)$ respectively.

Lemma 4.28. $\psi(\alpha)_\lambda = \psi(\alpha_\lambda)$

Proof. Let $\sigma_{\lambda_1} \in \mathrm{H}^0(S_\lambda,\Omega^2_{S_\lambda})$ be a differential form on S_λ. As ψ is Hodge $\sigma_{\lambda_2} := \psi(\sigma_{\lambda_1})$ is a differential form on T_λ. Let $V_\lambda = \langle \alpha_\lambda, \operatorname{Re}\sigma_{\lambda_1}, \operatorname{im}\sigma_{\lambda_1}\rangle$ and $W_\lambda = \langle \psi(\alpha)_\lambda, \operatorname{Re}\sigma_{\lambda_2}, \operatorname{im}\sigma_{\lambda_2}\rangle$.

As proven in Corollary 4.18 the vector spaces $V := V_\lambda$, $W := W_\lambda$ are constant with respect to λ. Hence as shown in Remark 4.23, ψ restricts to an isometry $V \to W$.

Since $\psi : V \to W$ is an isometry, and the $(1,1)$-forms are orthogonal to the differential forms we must have $\psi(\alpha_\lambda) = c_\lambda \psi(\alpha)_\lambda$ for some non-zero $c_\lambda \in \mathbb{R}$. We cited in Remark 4.17 that the norms of the Kähler classes remain constant as they deform. Thus $\|\alpha_\lambda\| = \|\alpha\| = \|\psi(\alpha)\| = \|\psi(\alpha)_\lambda\|$.

This forces $c_\lambda^2 = 1$. Since the deformations are continuous, so must c_λ vary continuously. Hence c_λ assumes its original value throughout, which is 1. □

Corollary 4.29. *Image of the map $\mathbb{P}_{\psi,\alpha} \to \mathcal{M} \times \mathcal{M}$ lies in $\mathcal{M}_\phi$.*

Proof. Definition 3.1 states two conditions that need to be checked on ψ_λ. The first condition requires ψ_λ to be a Hodge isometry, which is the statement of Theorem 4.25. The second condition requires the Kähler cones to intersect and this is implied by the lemma above since α_λ and $\psi(\alpha)_\lambda$ are both Kähler, while ψ_λ maps one to the other. □

Definition 4.30. Let $\alpha \in \mathrm{H}^{1,1}(S,\mathbb{R})$ be a positive class on S, which is not necessarily Kähler. Denote the graph of $\phi : Q_{S,\alpha} \to Q_{T,\psi(\alpha)}$ in Ω_ϕ by $Q_{\psi,\alpha}$. Then $Q_{\psi,\alpha}$ is called a *twistor line* in the twisted period domain Ω_ϕ.

Remark 4.31. We have shown that if $\alpha \in K_S \cap \psi^{-1}(K_T)$ then $Q_{\psi,\alpha}$ admits a lift to $\mathcal{M}_\phi$, namely $\mathbb{P}_{\psi,\alpha}$.

5. Twistor connectivity of $\mathcal{M}_\phi$

Definition 5.1. A K3 surface is *generic* if it has trivial Picard group. A point of $\mathcal{M}$, $\mathcal{M}_\phi$, Ω_Λ or Ω_ϕ corresponding to a generic K3 (or a pair of generic K3s) will be called a *generic point*. Any twistor path containing a generic point will be called a *generic twistor path*.

Lemma 5.2. *If S is a generic K3 then any positive $\alpha \in \mathrm{H}^{1,1}(S,\mathbb{R})$ is Kähler.*

Proof. See Proposition 7.3.7 of [Huy] and the reference therein. □

Corollary 5.3. *Let (S,η) be a generic marked K3. Then any twistor line through $x := \pi(S,\eta) \in \Omega_\Lambda$ is Kähler.*

Proof. We showed in Remark 4.5 that any twistor line Q through x is of the form $Q_{S,\alpha}$ for some positive $(1,1)$-class α on S. This class α is Kähler by Lemma 5.2. □

Corollary 5.4. *Any generic twistor path in Ω_Λ with generic end points lifts to a connected path in $\mathcal{M}^+$ (or in $\mathcal{M}^-$) connecting the unique lifts of the end points.*

Proof. The previous corollary allows for the existence of lifts as stated in Remark 4.21. Injectivity of π over the periods of generic K3s ensure that the path is connected. □

Corollary 5.5. *Any two points in Ω_ϕ are connected by a generic twistor path.*

Proof. Given $x = (x', x''), y = (y', y'') \in \Omega_\phi$, find a generic twistor path in Ω_Λ connecting x' to y'. The graph of this path lies in Ω_ϕ (see Remark 3.7) and connects x to y. □

Remark 5.6. There is an isomorphism $\mathcal{M}_\phi^+ \to \mathcal{M}_\phi^-$ established by sending (S,η,T,μ) to $(S,-\eta,T,-\mu)$. In particular, this means that we only need to prove connectivity results for $\mathcal{M}_\phi^+$ and the analogous results will hold for $\mathcal{M}_\phi^-$.

Lemma 5.7. *Any twistor line in Ω_ϕ through a generic point lifts to a twistor line in $\mathcal{M}_\phi^+$.*

Proof. Let Q be a twistor path containing a generic point $x \in \Omega_\phi$. Let $(S,\eta,T,\mu) \in \mathcal{M}_\phi^+$ be the preimage of x and let $\psi = \mu^{-1} \circ \phi \circ \eta$. We may express Q as $Q_{\psi,\alpha}$ for some positive $\alpha \in \mathrm{H}^{1,1}(S,\mathbb{R})$. Since S and T are generic, by Lemma 5.2, both α and $\psi(\alpha)$ are Kähler classes. Consequently $\mathbb{P}_{\psi,\alpha}$ provides a lift of Q. □

Corollary 5.8. *Any two generic points of $\mathcal{M}_\phi^+$ are connected by a twistor path. (Similarly for $\mathcal{M}_\phi^-$.)*

Proof. Take generic $x, y \in \mathcal{M}_\phi^+$ and a generic twistor path through their images $\pi(x)$ and $\pi(y)$ in Ω_ϕ. As each component of the path contains generic points, we may lift each component to $\mathcal{M}_\phi^+$ as in Lemma 5.7. But $\mathcal{M}_\phi^+ \to \Omega_\phi$ is injective over generic points. As such the lift of the path is connected with end points x and y. □

Lemma 5.9. *Take an arbitrary point $(S,\eta,T,\mu) \in \mathcal{M}_\phi^+$, let $\psi = \mu^{-1} \circ \phi \circ \eta$. There exists a Kähler class $\alpha \in K_S \cap \psi^{-1}(K_T)$ such that the twistor line $\mathbb{P}_{\psi,\alpha}$ contains a generic point.*

Proof. For non-zero $l \in \Lambda$ denote by $H_l \subset \Lambda_{\mathbb{R}}$ the orthogonal hyperplane to l. A twistor line $Q_{S,\alpha}$ contains no generic elements precisely when $V_{S,\alpha}$ is contained in $\bigcup_{0\neq l\in\Lambda} H_l$. As $V_{S,\alpha}$ and H_l are all linear spaces, this can happen only if there exists an l such that $V_{S,\alpha} \subset H_l$.

Let $V' = \eta\left(\langle \operatorname{Re}\sigma_S, \operatorname{im}\sigma_S\rangle\right) \subset V_{S,\alpha}$ be the part independent of α. Let $\Lambda^{1,1} = \eta\left(\mathrm{H}^{1,1}(S,\mathbb{R})\right)$. If $V' \subset H_l$ then the inclusion $\Lambda^{1,1} \cap H_l \subsetneq \Lambda^{1,1}$ is proper (otherwise $H_l = \Lambda_{\mathbb{R}}$, which is absurd).

Define the union $H' = \bigcup_{V'\subset H_l} H_l$. This (at most) countable union of hyperplanes can not cover $\Lambda^{1,1}$ since they all properly intersect it. As such, H' can not cover the open cone $\eta(K_S \cap \psi^{-1}(K_T))$.

Pick any $\alpha \in K_S \cap \psi^{-1}(K_T)$ such that $\eta(\alpha) \notin H'$. Then $V_{S,\alpha} \not\subset H_l$ for any non-zero $l \in \Lambda$. Hence the lift $\mathbb{P}_{\psi,\alpha}$ contains a generic point. □

Theorem 5.10. *$\mathcal{M}_\phi^+$ is twistor connected (and so is $\mathcal{M}_\phi^-$).*

Proof. Let $x, y \in \mathcal{M}_\phi^+$. Using Lemma 5.9 we may connect x and y to generic points via twistor lines. Any two generic points of $\mathcal{M}_\phi^+$ are connected by a twistor path as stated in Corollary 5.8 and we are done. □

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