Enumerative Geometry of Double Spin Curves

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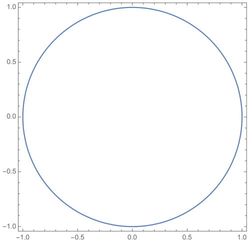
MAF
Moduli and Automorphic Forms

Introduction

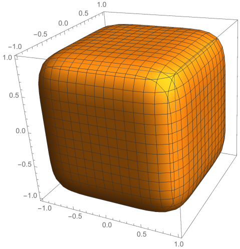
- Solving a polynomial: $x^2 + 2x + 2 = 0 \implies x \in \{1+i, 1-i\}$.
- Solving a system of polynomials:

$$\left. \begin{aligned} 2x^2 + 5x + 4y - y^2 - 6 = 0 \\ x + y = 0 \end{aligned} \right\} \implies (x, y) \in \{(-3, 3), (2, -2)\}$$

- Typically huge systems can not be solved.
- "How many solutions are there?" — a classification problem.
- What if there are infinitely many solutions?



(a) Dimension 1
$x^2 + y^2 - 1 = 0$



(b) Dimension 2
$x^6 + y^6 + z^6 - 1 = 0$

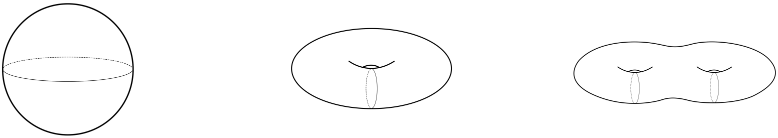
- First classify by dimension
- Then classify by the nature of the underlying shapes
- We will consider solutions over $\mathbb{C}$ — this doubles apparent dimension

Dimension 1 solution sets

Definition

Solution sets of polynomial systems over $\mathbb{C}$ of dimension 1 are called *complex curves*, provided they are smooth, proper and connected.

- Dimension 1 over $\mathbb{C}$ means dimension 2 over $\mathbb{R}$
- Classification of surfaces over $\mathbb{R}$ by genus



(a) genus 0 (b) genus 1 (c) genus 2

Moduli of curves

- Let $\mathcal{M}_g$ be the moduli space of complex curves of genus g
- The space $\mathcal{M}_g$ itself can be described by polynomials and

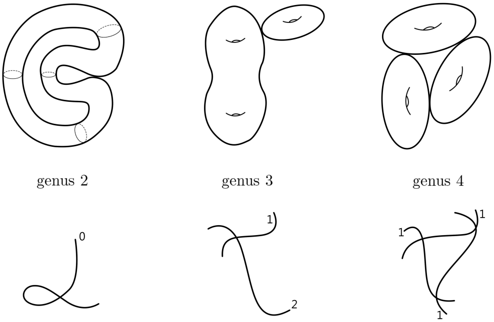
$$\dim_{\mathbb{C}} \mathcal{M}_g = \begin{cases} 3g-3 & g \geq 2 \\ 1 & g = 1 \\ 0 & g = 0 \end{cases}$$

- But $\mathcal{M}_g$ is not complete

Stable curves

Definition

A *stable curve* is a possibly singular complex curve with at worst nodes and finitely many automorphisms.



- Let $\overline{\mathcal{M}}_g$ be the moduli space of stable curves of genus g
- $\mathcal{M}_g \hookrightarrow \overline{\mathcal{M}}_g$

Theorem (Deligne–Mumford 1969)

The moduli space $\overline{\mathcal{M}}_g$ is irreducible, proper and smooth.

Section 2

Contact hyperplanes

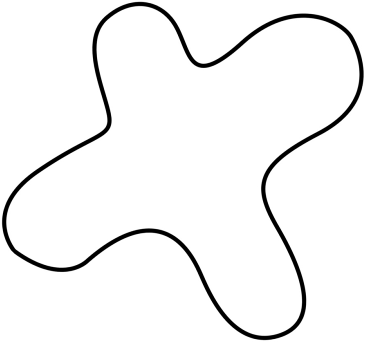
Canonical map

- Let C be a genus g curve,
- ω_C its cotangent bundle,
- $\varphi : C \to \mathbb{P}(\mathrm{H}^0(C, \omega_C)) \simeq \mathbb{P}^{g-1}$ the canonical map.
  - If C is not hyperelliptic then φ is an embedding,
  - otherwise, φ is a 2:1 cover of the rational normal curve.

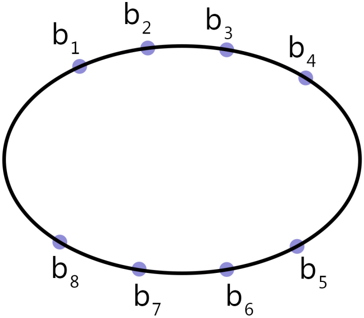
Theorem (Franchetta conjecture, Arbarello–Cornalba 1987)

The only consistent way to define a projective map on any family of curves is to use the canonical map or its powers.

Canonical curve of genus 3

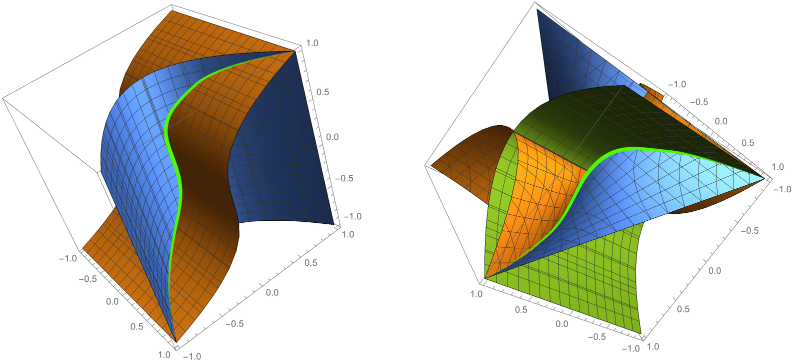


(a) genus 3, non-hyperelliptic



(b) genus 3, hyperelliptic

Canonical curve of genus 4



(a) genus 4, non-hyperelliptic

(b) genus 4, hyperelliptic

Contact hyperplanes

If $H \subset \mathbb{P}^{g-1}$ is a hyperplane then $\varphi^* H = H \cdot C$ is the intersection divisor of C with H.

Definition

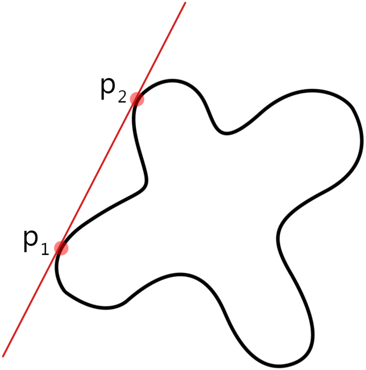
If $H \cdot C = 2D$ for an effective divisor D on C then H is called a *contact hyperplane* of C.

Definition

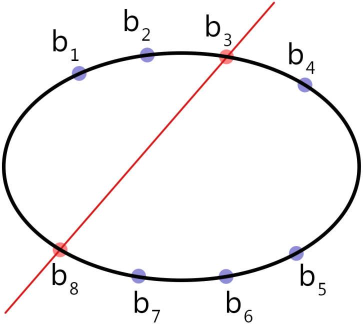
This D is called the *contact divisor* of H and any point $p \in \operatorname{supp}(D)$ is called a *contact point* of H.

- $\deg H \cdot C = \deg \omega_C = 2g - 2$,
- $\deg D = g - 1$.

Contact hyperplanes in genus 3



(a) genus 3, non-hyperelliptic



(b) genus 3, hyperelliptic

A contact hyperplane in genus 4

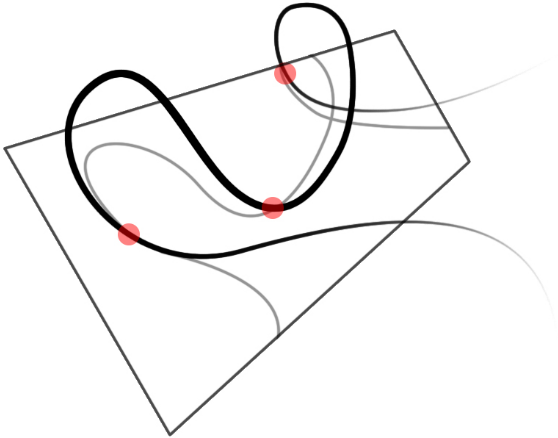


Figure: genus 4, non-hyperelliptic

Goal

Motivation: classical results and modern tools

- Bitangents to plane curves: Plücker, Steiner, Riemann
- Contact planes: Frobenius, Coble, Mumford, Harris
- Moduli: Cornalba, Farkas, Verra, Sernesi, Caporaso

Goal

Bring in degeneration techniques to the study of tuples of contact hyperplanes.

- Let $H_1, H_2 \subset \mathbb{P}^{g-1}$ be two contact hyperplanes of C
- Let $D_1, D_2 \subset C$ be the contact divisors of H_1 and H_2

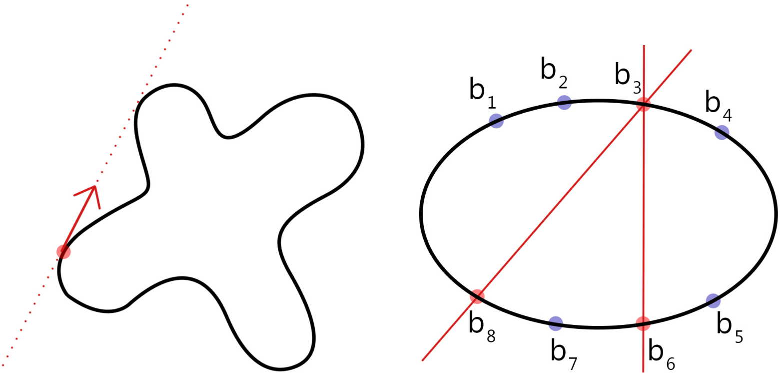
Definition

If $D_1 \cap D_2 \neq \emptyset$ then (H_1, H_2) is said to have *common contact* on C.

Definition

Then (H_1, H_2) will be called a *common contact pair*.

Common contact in low genus



(a) genus 3, non-hyperelliptic (b) genus 3, hyperelliptic

Guiding problems

Guiding Problem A

How "many" curves are there admitting common contact pairs?

Genus 3: Only when C is hyperelliptic, which is a divisorial condition in $\mathcal{M}_3$.

Guiding Problem B

If (H_1, H_2) has one point of common contact, does it have more?

Genus 3: No. Contact divisors have degree 2.

Guiding Problem C

When there is one, are there other common contact pairs (H_1, H_2)?

Genus 3: There are 336 ordered pairs (H_1, H_2) with common contact on hyperelliptic curves.

Section 3

Spin curves

Theta characteristics

Observation: If $H \cdot C = 2D$ then $\mathcal{O}_C(D)^{\otimes 2} \simeq \omega_C$.

Definition

A line bundle η on C such that $\eta^{\otimes 2} \simeq \omega_C$ is called a *theta characteristic*.

Definition

A tuple (C, η, α) where $\alpha : \eta^{\otimes 2} \xrightarrow{\sim} \omega_C$ is called a *spin curve*.

- If η is a theta characteristic and $D \in |\eta|$ then $2D \in |\omega_C|$,
  - that is, $\exists H \subset \mathbb{P}^{g-1}$ such that $H \cdot C = 2D$.
- There is a bijection of sets:

$$\{(\eta, D \in |\eta|) \mid \eta^{\otimes 2} \simeq \omega_C\} \leftrightarrow \{H \mid \text{contact hyperplanes of } C\}.$$

Parity

Definition

A theta characteristic η is said to be *even* or *odd* according to the parity of the integer $h^0(\eta) = \dim_{\mathbb{C}} \mathrm{H}^0(C, \eta)$.

Theorem (Folklore, Mumford 1971, Atiyah 1971)

Parity is a deformation invariant.

- Let $\mathcal{S}_g = \{(C, \eta) \mid C \in \mathcal{M}_g,\ \eta^{\otimes 2} \simeq \omega_C\}$ be the moduli space of spin curves.
  - Moduli of even spin curves: $\mathcal{S}_g^+ = \{(C, \eta) \in \mathcal{S}_g \mid h^0(\eta) \equiv 0 \pmod 2\}$,
  - Moduli of odd spin curves: $\mathcal{S}_g^- = \{(C, \eta) \in \mathcal{S}_g \mid h^0(\eta) \equiv 1 \pmod 2\}$.
- Deformation invariance of parity $\implies \mathcal{S}_g = \mathcal{S}_g^+ \sqcup \mathcal{S}_g^-$.

In fact, parity and genus are the *only* deformation invariants of a spin curve:

Theorem (Folklore, Cornalba 1989)

The moduli spaces $\mathcal{S}_g^+$ and $\mathcal{S}_g^-$ are irreducible for all $g \geq 1$.

Theorem (Folklore, Harris 1982)

If $(C, \eta) \in \mathcal{S}_g$ is general then $h^0(\eta)$ is minimal, i.e., $h^0(\eta) \in \{0, 1\}$.

- In general, even theta characteristics do not contribute to contact hyperplanes
- In general, odd theta characteristics contribute a unique contact hyperplane

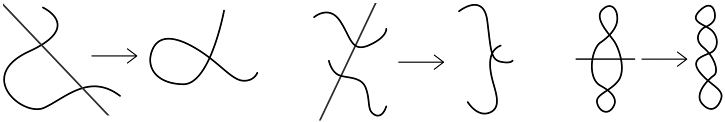
We therefore have the following simplified bijection on a generic curve C:

$$\{\eta \mid \text{odd theta characteristic}\} \leftrightarrow \{H \mid \text{contact hyperplanes of } C\}.$$

Quasi-stable curves

Definition

A curve Y is called *quasi-stable* if Y can be obtained by blowing-up a subset of the nodes of a stable curve X.



The components obtained during blow-up are unstable and are called *exceptional components*.

Stable spin curves

- Let Y be a quasi-stable curve of genus g.
- Let L be a line bundle on Y of degree $g-1$,
  - such that, $\deg L|_E = 1$ on every exceptional component E of Y.

Definition

A tuple (Y, L, σ) with (Y, L) as above is called a *stable spin curve* if $\sigma : L^{\otimes 2} \to \omega_Y$ is an isomorphism in the complement of the exceptional components.

These include:

- Spin curves.
- Stable curves with roots of the canonical bundle.

Compactifying moduli of spin curves

- Let $\overline{\mathcal{S}}_g$ be the moduli space of stable spin curves of genus g.

Theorem (Cornalba 1989, Jarvis 1998)

The moduli space $\overline{\mathcal{S}}_g$ is proper, contains $\mathcal{S}_g$ as a dense open subset and is smooth as a stack.

- Let $\pi : \overline{\mathcal{S}}_g \to \overline{\mathcal{M}}_g : (Y, L) \mapsto \overline{Y}$ be obtained by forgetting the spin structure and stabilizing the curve.
- The morphism π is finite between the coarse moduli spaces.

Section 4

Multiple spin curves

Moduli of multiple spin curves

Fix an integer $m \geq 2$.

Definition

A tuple $(C, \eta_1, \ldots, \eta_m)$ where $\eta_i^{\otimes 2} \simeq \omega_C$ is a *multiple spin curve*.

- The moduli space of multiple spin curves can be constructed simply as a product: $\mathcal{S}_g^m = \mathcal{S}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{S}_g$.
- Compactify $\mathcal{S}_g^m$ as follows:

$$\overline{\mathcal{S}}_g^{\times m} = \overline{\mathcal{S}}_g \times_{\overline{\mathcal{M}}_g} \cdots \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g.$$

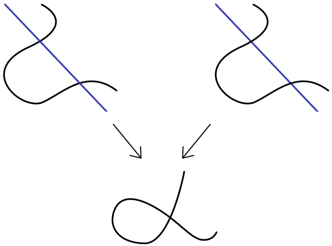
Trouble:

- The objects parametrized by $\overline{\mathcal{S}}_g^{\times m}$ are not even curves with m-line bundles!
- The space $\overline{\mathcal{S}}_g^{\times m}$ is non-normal.

- Let $m = 2$ and consider $\overline{\mathcal{S}}_g^{\times 2} = \overline{\mathcal{S}}_g \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{S}}_g$.
- Product consists of pairs with identified image:

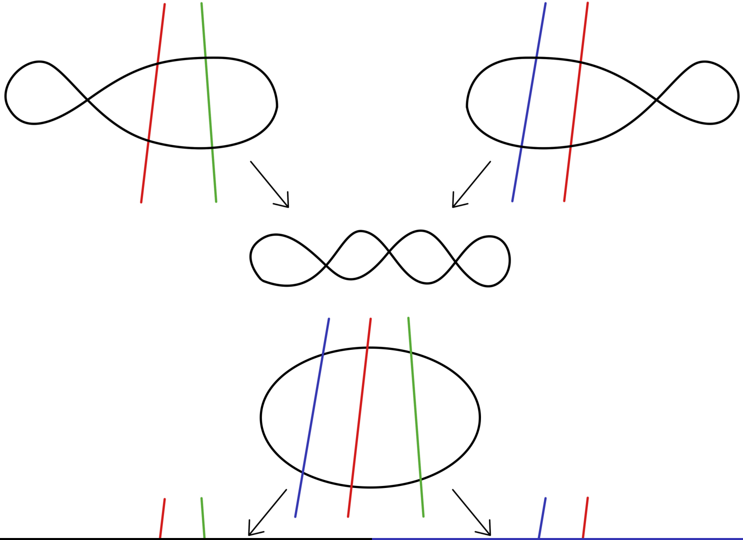
$$((Y_1, L_1), (Y_2, L_2), f : \overline{Y}_1 \overset{\sim}{\to} \overline{Y}_2) \in \overline{\mathcal{S}}_g^{\times 2}$$

- Two line bundles on two different curves:



- **Idea:** Introduce an isomorphism $\iota : Y_1 \overset{\sim}{\to} Y_2$.
- **Issue:** Infinite fibers.
- **Idea:** Require $L_1^{\otimes 2} \simeq \iota^* L_2^{\otimes 2}$.

A more complicated example



Solution

$$
\begin{array}{ccccc}
 & & Y & & \\
 & \overset{\pi_1}{\swarrow} & & \overset{\pi_2}{\searrow} & \\
Y_1 & & & & Y_2 \\
 & \searrow & & \swarrow & \\
 & & X & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & \mathcal{F} & & \\
 & \overset{\rho_1}{\swarrow} & & \overset{\rho_2}{\searrow} & \\
\pi_1^* L_1^{\otimes 2} & & & & \pi_2^* L_2^{\otimes 2} \\
 & \searrow & & \swarrow & \\
 & & \omega_Y & &
\end{array}
$$

Stable multiple spin curves

- Y a quasi-stable curve
- $L_1, \ldots, L_m$ line bundles on Y
- each L_i is the pullback of a spin structure from a partial stabilization
- each pair $L_i^{\otimes 2}$ and $L_j^{\otimes 2}$ are isomorphic away from certain exceptional divisors
- for each exceptional component $E \subset Y$ there is at least one L_i with $\deg L_i|_E = 1$

Definition

We will call such a tuple $(Y, L_1, \ldots, L_m)$, taken together with the relevant squaring maps, a *stable multiple spin curve*.

Moduli of multiple spin curves

Let $\overline{\mathcal{S}}_g^m$ be the moduli space of stable multiple spin curves.

Theorem (—, 2017)

The moduli space $\overline{\mathcal{S}}_g^m$ *for* $g \geq 2$ *and* $m \geq 1$ *is a proper smooth Deligne–Mumford stack, containing* $\mathcal{S}_g^m$ *as a dense open substack.*

Remark

The forgetful functor $\overline{\mathcal{S}}_g^m \to \overline{\mathcal{M}}_g$ is finite between the coarse moduli spaces.

Classifying the components

- **Recall:** $\overline{\mathcal{S}}_g = \overline{\mathcal{S}}_g^- \sqcup \overline{\mathcal{S}}_g^+$; both components are irreducible.
- **Goal:** Find all irreducible components of $\overline{\mathcal{S}}_g^m$.
- Redundancies are inevitable. For instance $\overline{\mathcal{S}}_g^2$ has a "diagonal" component. Such components are called *degenerate*.
- We need a new invariant:

Definition

A triplet of theta characteristics η_1, η_2, η_3 are said to be *syzygetic* if $\omega_C^{\otimes 2} \otimes \eta_1^\vee \otimes \eta_2^\vee \otimes \eta_3^\vee$ is an *odd* theta characteristic. The triplet is *asyzygetic* otherwise.

Theorem (—,2017)

For $g \geq m$, the space $\overline{\mathcal{S}}_g^m$ has precisely $2^{m+\binom{m-1}{2}}$ non-degenerate irreducible components. They are determined solely by the parity of each spin and the syzygy relations of subtriplets.

Remark

When $g < m$ there are strictly fewer components. But there is an effective algorithm to find the "missing" components.

- $\overline{\mathcal{S}}_g^+, \overline{\mathcal{S}}_g^-$ are irreducible
- $\overline{\mathcal{S}}_g^{++}, \overline{\mathcal{S}}_g^{+-}, \overline{\mathcal{S}}_g^{-+}, \overline{\mathcal{S}}_g^{--} \subset \overline{\mathcal{S}}_g^2$ are irreducible
- $\overline{\mathcal{S}}_g^{---} = \overline{\mathcal{S}}_g^{syz} \sqcup \overline{\mathcal{S}}_g^{asy} \subset \overline{\mathcal{S}}_g^3$ has two irreducible components

Section 5

Implications

How often do common contact pairs appear?

- $\overline{\mathcal{S}}_g^{--} \subset \overline{\mathcal{S}}_g^{\times 2}$: moduli of double odd spin curves
- We are primarily interested in the locus:

$$\Omega_g := \{(C, \eta_1, \eta_2) \mid \eta_1 \cap \eta_2 \neq \emptyset\} \subset \mathcal{S}_g^{--},$$

- and its Zariski closure $\overline{\Omega}_g \subset \overline{\mathcal{S}}_g^{--}$.

Lemma

The locus $\overline{\Omega}_g$ is pure of codimension 1 in $\overline{\mathcal{S}}_g^{--}$.

Guiding Problem A

Compute the divisor class $[\overline{\Omega}_g] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$.

We have standard divisor classes on $\overline{\mathcal{S}}_g^{--}$ obtained from:

- The boundary components:

$$\overline{\mathcal{S}}_g^{--} \setminus \mathcal{S}_g^{--} = \Delta_0^{bb} \cup \Delta_0^{b=} \cup \Delta_0^{bn} \cup \Delta_0^{nb} \cup \Delta_0^{nn} \bigcup_{i=1}^{g-1} \left(\Delta_i^{++} \cup \Delta_i^{+-} \cup \Delta_i^{+=} \cup \Delta_i^{-=}\right).$$

- The Hodge class λ
  - measures the twisting of the canonical map in families.

Let $\delta_i^{xy} = [\Delta_i^{xy}] \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$.

Theorem (—,2017)

If $\overline{\Omega}_g$ is irreducible, in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{S}}_g^{--})$ we have:

$$[\overline{\Omega}_g] = \frac{g+5}{2}\lambda - \frac{g+1}{8}\delta_0^{nn} - \frac{g+3}{8}\left(\delta_0^{nb} + \delta_0^{bn}\right) - \delta_0^{bb} - (g-1)\delta_0^{b=} - \sum_{i=1}^{g-1}\left((2i-1)\delta_i^{++} - (g-1)\delta_i^{+-} - (3i-1)\delta_i^{+=} - (g+i-2)\delta_{g-i}^{-=}\right)$$

Corollary

For $g \geq 10$ the canonical class of $\overline{\mathcal{S}}_g^{--}$ is big. Hence, if $\overline{\mathcal{S}}_g^{--}$ has mild singularities then it is of general type in this range.

Number of common contact points of a given pair

Guiding Problem B

How many common contact points does a generic pair in $\overline{\Omega}_g$ have?

Lemma

If $\overline{\Omega}_g$ is irreducible then the generic element of $\overline{\Omega}_g$ has a unique common contact point.

Number of common contact pairs

Guiding Problem C

What is the degree of $\overline{\Omega}_g \subset \overline{\mathcal{S}}_g^{--}$ over its image in $\overline{\mathcal{M}}_g$?

- Recall that this degree is 336 when $g = 3$.
- Expected degree for $g \geq 4$ is 2 (which is minimal).

Strategy:

- $\pi : \overline{\mathcal{S}}_g^{--} \to \overline{\mathcal{M}}_g$
- For $X \in \overline{\mathcal{M}}_g$ let $f_X = \#(\pi^{-1}(X) \cap \overline{\Omega}_g)$
- Find one X for which $f_X = 2$
- Pick $X \in \Delta_0 \subset \overline{\mathcal{M}}_g$ and break f_X into smaller pieces:
- $f_X = f_X^{bb} + f_X^{b=} + f_X^{bn} + f_X^{nb} + f_X^{nn}$

Theorem (—,2017)

When $g \geq 4$ there exists a curve $X \in \overline{\mathcal{M}}_g$ such that $f_X^{bb} = 2$ and $f_X^{b=} = f_X^{bn} = f_X^{nb} = 0$.

Lemma

For the same X, we have $f_X^{nn} = 0$ provided that there exists a hyperelliptic curve C of genus $g-1$ and two Weierstrass points $w_1, w_2 \in C$ such that any distinct pair of roots $\tau_1, \tau_2 \in \sqrt{\omega_C(w_1 + w_2)}$ has disjoint zero divisors.

Corollary

If $\overline{\Omega}_g$ is irreducible and the hypothesis of the lemma above is satisfied then $\overline{\Omega}_g$ is of degree 2 over its image in $\overline{\mathcal{M}}_g$.

Summarizing guiding problems B and C

- Let $g \geq 4$, $(C, \eta_1, \eta_2) \in \Omega_g$
- Let $H_i \subset \mathbb{P}^{g-1}$ be the contact hyperplanes of C corresponding to η_i

With the hypotheses in the previous slide:

GP B H_1 and H_2 have a unique point of common contact.

GP C C admits no other common contact pairs besides (H_1, H_2).

Thank you!