Deformation Theory

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Lecture 3

1 Category theory and families of objects

1.1 Yoneda's Lemma

Recall that a functor F is contravariant if an arrow $A \to B$ is mapped to an arrow $F(B) \to F(A)$; in other words, if F is an 'arrow reversing functor'.

Let $\mathcal{C}$ be a category. Yoneda's lemma lets you view an object $X \in \mathcal{C}$ as the "same thing" as a contravariant functor

$$h_X : \mathcal{C} \to (\text{Sets}),$$

where h_X is defined by

$$h_X(Y) = \hom_{\mathcal{C}}(Y, X).$$

If $f : X \to Y$ is a morphism and $U \in \mathcal{C}$ an object, we can define a function

$$\begin{array}{cccccc} h_f(U) & : & h_X(U) & \to & h_Y(U) & \\ & & \| & & \| & , \\ & & \hom(U,X) & & \hom(U,Y) & \end{array}$$

via pre-composition with f. In fact, h_f is a morphism of functors $h_X \to h_Y$.

We may now state the weak version of Yoneda's lemma.

Lemma 1.1.1 (Yoneda's Lemma)**.** *With $X, Y \in \mathcal{C}$ the function just defined:*

$$\hom_{\mathcal{C}}(X, Y) \to \hom(h_X, h_Y) : f \mapsto h_f$$

is bijective.

Exercise. Prove the lemma.

1.2 Representability

Now suppose one wants to study families of some object, such as schemes or line bundles. One starts by defining an appropriate contravariant functor

$$F : \text{Sch}_k \to (\text{Sets}).$$

One considers, in a loose sense, F as parametrizing the objects in the set $\Lambda := F(\text{Spec}(k))$.

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Indeed, for any scheme S the set $F(S)$ may be regarded as all possible families over S of objects Λ. Therefore F is a machinery telling how the elements in Λ can vary.

Now We want to have a notion of what it means for a *scheme* Z to parametrize the objects in the set Λ.

The strongest interpretation of parametrizing $F(\mathrm{Spec}(k))$ is via the notion of representability.

Definition 1.2.1. Let Z be a scheme. We say the functor F is *represented by* Z if the functors F and h_Z are isomorphic.

Important Exercise. Show that F is represented by Z iff there exists a universal object $\xi \in F(Z)$ (necessarily unique) such that for any $\xi' \in F(Y)$ there exists a unique morphism $f_{\xi'} : Y \to Z$ satisfying $F(f_{\xi'})(\xi) = \xi'$. This object ξ is often referred to as the *universal family over* Z.

In the following section we illustrate these concepts by considering families of line bundles on a fixed scheme.

1.3 Families of line bundles

Let X be a smooth, projective variety over a field k. Let S be a scheme over k. Denote by X_S the fiber product $X \times_{\mathrm{Spec}(k)} S$ and let $\pi_S : X_S \to S$ be the second projection. The set $\mathrm{Pic}(X_S)$ is the set of isomorphism classes of line bundles on X_S.

Note that a line bundle $\mathcal{L}$ on X_S can be regarded as a family of line bundles on X parametrized by S. Indeed, if $x \in S$ is a closed point then the fiber of $X_S \to S$ over x is isomorphic to X and by pulling back $\mathcal{L}$ to this fiber we get a line bundle on X.

Exercise. Check that if $\mathcal{N}$ is a line bundle on S then $\mathcal{L} \otimes \pi_S^*(\mathcal{N})$ and $\mathcal{L}$ give the same family of line bundles on X. That is, on each fiber the pullbacks of these line bundles coincide.

To correct this problem observe that $\mathrm{Pic}(X_S)$ is an abelian group and that the pullback by π_S gives a group morphism $\mathrm{Pic}(S) \to \mathrm{Pic}(X_S)$ so that we may define a new functor $\mathfrak{Pic}$ as follows:

$$\mathfrak{Pic} : \mathrm{Sch}_k \to (\mathrm{Sets}) : S \mapsto \mathrm{Pic}(X_S)/\,\mathrm{Pic}(S).$$

Still $\mathfrak{Pic}(\mathrm{Spec}(k))$ is equal to the set of isomorphism classes of line bundles on X. Moreover, $\mathfrak{Pic}$ appears to be a better functor in parametrizing line bundles on X.

If $\mathfrak{Pic}$ is representable then there exists a scheme T over k, together with a 'universal' line bundle $\mathcal{L} \in \mathrm{Pic}(X \times T)$ in the following sense. For any scheme S over k and line bundle $\mathcal{M} \in \mathrm{Pic}(X_S)$ there exists a unique morphism $\varphi : S \to T$ with $(\varphi \times \mathrm{Id}_X)^*(\mathcal{L})$ isomorphic to $\mathcal{M}$ modulo $\mathrm{Pic}(S)$.

Whether some functor of families of geometric objects is representable is in general a very difficult question to answer. The first step in this direction is often to restrict the functor to Artin rings.

2 Artin Rings

2.1 Motivation

In the previous lecture we saw that the deformation theory of associative algebras is done in a step-by step fashion. To be precise, we started with a differential $F_1 \in H^2(A, A)$ and we wanted to find a 1-parameter family of deformations of A of the form:

$$f_t(a, b) = ab + tF_1(a, b) + t^2F_2(a, b) + \dots$$

The procedure went like this:

We used F_1 to produce an appropriate F_2, assuming the associator of F_1 is equivalent to 0 in $H^3(A, A)$.

When we have $F_1, \dots, F_{n-1}$ we produced F_n under a certain condition on the F_i's (this time $[G_n] \in H^3(A, A)$ must be 0).

Then by induction we get all the terms. We now want to formalize this procedure.

2.2 Definition of an Artin ring

Let k be a fixed field. Let A be a commutative algebra over k (with unit). We say A is *Artinian* if it satisfies the descending chain condition on ideals, ie. if any descending sequence of ideals

$$\dots \subset I_4 \subset I_3 \subset I_2 \subset I_1 \subset A$$

eventually stabilizes. Stability here means that there is an n_0 such that for all $n \geq n_0$ we have $I_n = I_{n_0}$.

Examples. The k-algebra $k[t]/(t^n)$ is Artinian. An integral domain is Artinian if and only if it is a field.

Recall that a commutative ring R is *local* if it has a unique maximal ideal. If R is a local ring with maximal ideal $\mathfrak{m}$, then the field $R/\mathfrak{m}$ is the *residue field.*

If A and B are local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively then a morphism $\varphi : A \to B$ is said to be a *local morphism* if $\varphi(\mathfrak{m}_A) \subset \mathfrak{m}_B$.

Notation. Let Art_k be the category whose objects are local Artinian k-algebras with residue field k and whose arrows are local morphisms.

Given $A \in \mathrm{Art}_k$ we define $r_A : \mathrm{Spec}(k) \to A$ to be the map associated to the quotient $A \to A/\mathfrak{m}_A \simeq k$. Note the isomorphism of the quotient field with the residue field is canonically induced via the structure map $k \to A$.

For the next few weeks, the main objects of study are (covariant/ordinary) functors

$$F : \mathrm{Art}_k \to (Sets)$$

such that $F(k)$ is a set with a single element.

2.3 Back to line bundles

Let $\mathfrak{Pic} : \mathrm{Sch}_k \to (\mathrm{Sets})$ be the contravariant functor defined in Section 1.3. Pick $\mathcal{M} \in \mathrm{Pic}(\mathrm{Spec}(k))$, ie. $\mathcal{M}$ is a line bundle on X.

We define a *covariant* functor

$$\mathfrak{Pic}_{\mathcal{M}} : \mathrm{Art}_k \to (\mathrm{Sets})$$

by

$$\mathfrak{Pic}_{\mathcal{M}}(A) := \{N \in \mathrm{Pic}(\mathrm{Spec}(A)) \mid r_A^* N \simeq \mathcal{M}\}$$

where $r_A : X_k \to X_A$ is induced from $r_A : \mathrm{Spec}(k) \to \mathrm{Spec}(A)$ in the obvious way. Note that $\mathfrak{Pic}_{\mathcal{M}}(k) = \{\mathcal{M}\}$. Loosely speaking, $\mathfrak{Pic}_{\mathcal{M}}$ is the functor of deformations of the line bundle $\mathcal{M}$.

In a similar fashion, from a contravariant functor $F : \mathrm{Sch}_k \to (\mathrm{Sets})$ and an object $a \in F(k)$ we can define a covariant functor $F_a : \mathrm{Art}_k \to (\mathrm{Sets})$ by restriction.

2.4 Tangent space

Define $k[\varepsilon] := k[x]/(x^2)$ to be the dual numbers. Recall, if X is a variety over k with a chosen point $x \in X$ (that is, $x \in \hom(\mathrm{Spec}(k), X)$), then the Zariski tangent space of X at x could be identified with the subset of $\hom(\mathrm{Spec}(k[\varepsilon]), X)$ such that the unique point in the spectrum of the dual numbers hits x.

Generalizing this definition we get:

Definition 2.4.1 (Schlessinger)**.** Let $F : \mathrm{Art}_k \to (\mathrm{Sets})$ be a functor such that $F(k)$ is a one-element set. Then $F(k[\varepsilon])$ is called the *tangent space* of F.

3 Completions of local rings

3.1 Definitions

We will recall the definition of inverse limits. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of groups together with homomorphisms $\theta_{n+1} : A_{n+1} \to A_n$. The *inverse limit* $\varprojlim A_n$ is the group of all sequences $\{a_n \mid a_n \in A_n, \theta_{n+1}(a_{n+1}) = a_n\}$.

The inverse limit comes with obvious projections:

$$\pi_i : \varprojlim A_n \to A_i.$$

The pair $(\varprojlim A_n, \{\pi_i \mid i \in \mathbb{N}\})$ is universal amongst such objects. See Section 3.2 for a discussion of this notion.

If R is a commutative ring and $I \subset R$ is a proper ideal, we get a commutative ring

$$\hat{R}_I := \varprojlim (R/I^n).$$

If R is a *local* commutative ring with maximal ideal $\mathfrak{m}$ we set

$$\hat{R} := \hat{R}_{\mathfrak{m}}.$$

For example:

- If $R = k[x_1, \ldots, x_n]$ and $I = (x_1, \ldots, x_n)$ then $\hat{R}_I = k[[x_1, \ldots, x_n]]$ is the ring of formal power series.

- If $R = \mathbb{Z}$ and $I = (p)$ for some prime $p \in \mathbb{Z}$ then $\hat{R}_I$ is the ring of p-adic integers.

More generally, if M is an R-module and $I \subset R$ is a proper ideal then we define

$$\hat{M}_I := \varprojlim M/I^n M,$$

which is a $\hat{R}_I$-module.

From a ring to its completion there is a natural map taking $a \in R$ to $(\bar{a} \in R/I^n)_{n\in\mathbb{N}}$. With this in mind we make the following definition.

Definition 3.1.1. Let R be a commutative ring and $I \subset R$ a proper ideal. We say R is *complete* with respect to I if the natural morphism $\varphi : R \to \hat{R}_I$ is an isomorphism.

Exercise. Let R be a local ring, $I \subset R$ a proper ideal. Then $\hat{R}_I$ is complete with respect to the ideal generated by $\varphi(I)$.

Notation. We let $\hat{\mathrm{Art}}_k$ denote the category of complete, noetherian local k-algebras R such that

$$R_n := R/\mathfrak{m}_R^n$$

is in Art_k for all $n \in \mathbb{N}$.

Exercise. Show Art_k is a subcategory of $\hat{\mathrm{Art}}_k$.

Exercise. Let $A \in \mathrm{Art}_k$ with $\mathfrak{m}_X^n = 0$ for some n. Pick $R \in \hat{\mathrm{Art}}_k$ and show that $\hom(R, A) = \hom(R_n, A)$.

Definition 3.1.2. (Main definition of this lecture) A functor $F : \mathrm{Art}_k \to (\mathrm{Sets})$ is called *pro-representable* if for some $R \in \hat{\mathrm{Art}}_k$ the functor F is isomorphic to the functor $\hom(R, \_)$ restricted to Art_k.

3.2 Digression on the universal property

It is often said that an object X is universal with respect to a property P. What this means is that the property P defines a category of objects $\mathcal{C}$ and the object X is either the terminal or the initial object in that category (to be understood from context). This interpretation makes it obvious that if a universal object exists (with respect to a property P) then it must be unique up to a unique isomorphism.

In defining the inverse limit of a sequence of groups let us see how this interpretation works. Fix $\{A_n\}_{n\in\mathbb{N}}$ and the sequence of morphisms $\theta = \{\theta_n : A_n \to A_{n-1}\}$. A careful reading of the universal property of the inverse limit suggests that we should build the category $\mathcal{C}$ whose objects are pairs (B, φ) where B is a group and $\varphi = \{\varphi_n : B \to A_n \mid n \in \mathbb{N}, \varphi_{n-1} = \theta_n \circ \varphi_n\}$ is a sequence of maps compatible with θ.

We are claiming that $(\varprojlim A_n, \pi)$ is the terminal object of this category $\mathcal{C}$. This is left as an exercise.

A nicer interpretation of the inverse limit in this case is via the theory of sheaves. Consider $\mathbb{N}$ with the following topology:

$$T = \{\emptyset\} \cup \{\mathbb{N}\} \cup \{U_n := \{0, \ldots, n\} \mid n \in \mathbb{N}\}.$$

Let $\mathcal{C}'$ be the subcategory of presheaves of groups on $(\mathbb{N}, T)$ consisting of presheaves $\mathcal{F}$ such that $\mathcal{F}(U_n) = A_n$ and $\mathcal{F}(U_{n-1} \hookrightarrow U_n) = \theta_n$. We have an equivalence of categories $\gamma : \mathcal{C}' \to \mathcal{C}$ defined as follows:

$$\gamma(\mathcal{F}) = (\mathcal{F}(\mathbb{N}), \{\mathcal{F}(U_n \hookrightarrow \mathbb{N}) \mid n \in \mathbb{N}\}).$$

The terminal object in $\mathcal{C}'$ is the one and only *sheaf* in $\mathcal{C}'$, and it is the sheafification of any $\mathcal{F}$ in $\mathcal{C}'$.

Exercise. Go to the 'Important Exercise' and identify the category of which the universal property identifies the terminal object.

Deformation Theory

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Lecture 4

We follow closely the following 2 papers:

- Fantechi, Manetti - "Obstruction Calculus for Functors of Artin Rings, I"
- Schlessinger - "Functors of Artin rings"

1 Extensions

Recall that Art_k is the category of local Artinian k-algebras with residue field k and $\hat{\mathrm{Art}}_k$ is the category of complete Noetherian local k-algebras such that for every $n \in \mathbb{N}$, $R_n = R/\mathfrak{m}_R^n$ is in Art_k.

Definition 1.1. A *small extension* in Art_k (respectively $\hat{\mathrm{Art}}_k$) is a short exact sequence

$$e : 0 \to M \to B \to A \to 0$$

where $B \to A$ is in Art_k (resp. in $\hat{\mathrm{Art}}_k$) and $\mathfrak{m}_B M = 0$. If in addition M is a principal ideal of B then we say e is *principal*.

Exercise. Suppose e is a small extension. Show that M is a k-vector space. Show further that e is principal iff $\dim_k M = 1$.

Notation. Let e be a small extension as in Definition 1.1. Then M, B and A are referred to as the *kernel, source* and *target*. We will also refer to them as $K(e), S(e)$ and $T(e)$ respectively.

Let Fvsp denote the category of finite dimensional k-vector spaces. If $A \in \hat{\mathrm{Art}}_k$ and $M \in \mathrm{Fvsp}$ then we define $\mathrm{Ex}(A, M)$ to be the set of isomorphism classes if extensions e with $K(e) = M$ and $T(e) = A$. This set $\mathrm{Ex}(A, M)$ carries a natural k-vector space structure. (This is because extensions are identified with $H^2(A, M)$ where M has the A module structure induced by $A \to k$. This cohomology group obviously has a k-vector space structure.)

Let $f : M \to N$ be a morphism in Fvsp. Then there is a linear map

$$f_* : \mathrm{Ex}(A, M) \to \mathrm{Ex}(A, N)$$

defined as follows: Let $e \in \mathrm{Ex}(A, M)$ then we have

$$\begin{array}{ccccccccc} 0 & \to & M & \to & B & \to & A & \to & 0 \\ & & \downarrow & & & & & & \\ & & N & & & & & & \end{array}$$

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and we will push forward the first exact sequence into another one as follows.

Set $B' = B \oplus N$. We make B' into a ring by defining:

$$(b_1, n_1) \cdot (b_2, n_2) = (b_1 b_2, \bar{b}_1 n_2 + \bar{b}_2 n_1)$$

where by $\bar{b}_i$ we mean the residue of b_i mod $\mathfrak{m}_B$, which is an element of k.

Then $B \in \hat{\mathrm{Art}}_k$ implies $B' \in \hat{\mathrm{Art}}_k$. Let $J \subseteq B'$ be the set

$$\{(m, f(m)) \mid m \in M \subset B\}.$$

It is easy to show that J is an ideal of B.

Let $\overline{B}$ denote B'/J. This gives us the pushforward $f_*(e)$ as follows:

$$f_*(e) : 0 \to N \to \overline{B} \to A \to 0$$

which lies in $\mathrm{Ex}(A, N)$.

Exercise. Let $g : A \to B$ be a morphism in $\hat{\mathrm{Art}}_k$. Show that this induces a linear map $g_* : \mathrm{Ex}(A, M) \to \mathrm{Ex}(B, M)$ satisfying the following identity

$$f_* g^* = g^* f_*$$

for any $f : M \to N$ in Fvsp.

Definition 1.2. A small extension e is called *trivial* if it splits (or, equivalently, if it corresponds to $0 \in \mathrm{Ex}(A, M)$).

We will define a morphism between two small extensions, thereby making a category out of small extensions. Let

$$e_i : 0 \to M_i \to B_i \to A_i \to 0$$

for $i = 1, 2$ be two small extensions.

A morphism $f : e_1 \to e_2$ is given by the data of a commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & M_1 & \to & B_1 & \to & A_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & M_2 & \to & B_2 & \to & A_2 & \to & 0 \end{array} .$$

2 Schlessinger's Conditions

Setup. Throughout the section $F : \mathrm{Art}_k \to (\mathrm{Sets})$ will be a functor such that $F(k)$ consists of a single element. We will let $\mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ denote the category of such functors.

Recall that F is called *pro-representable* if there exists a *complete* local k-algebra $R \in \hat{\mathrm{Art}}_k$ such that F is equivalent to $h_R|_{\mathrm{Art}_k}$. From now on we will not mention when h_R has to be restricted to Art_k, it is assumed to be understood from context.

Exercise. With F as in setup, show that for any $R \in \hat{\mathrm{Art}}_k$ there is a bijection

$$\varprojlim F(R/\mathfrak{m}_R^n) \simeq \hom(h_R, F)$$

where the hom on the RHS is the morphism of functors on Art_k. Note also that inverse limits of sets are perfectly well defined, as in abelian groups. [*Hint:* To construct the bijection notice that if $A \in \mathrm{Art}_k$ then any morphism $R \to A$ factors through $R/\mathfrak{m}_R^n$ for some n.]

Definition 2.1. Let $R \in \hat{\mathrm{Art}}_k$ and choose $\xi \in \varprojlim F(R/\mathfrak{m}_R^n)$. By the exercise above ξ corresponds to a morphism $h_R \to F$. We call such a pair (R, ξ) a *pro-couple.*

If F is pro-representable and (R, ξ) is a pro-couple corresponding to the isomorphism $h_R \to F$ then we say the pro-couple (R, ξ) *pro-represents* F.

Let $F \to G$ be a morphism in $\mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ and let $A \to B$ be a morphism in Art_k. We have a commutative diagram:

$$\begin{array}{ccc} F(A) & \to & F(B) \\ \downarrow & & \downarrow \\ G(A) & \to & G(B) \end{array}.$$

Thus we get, by the universal property of fiber products, a morphism

$$F(A) \to F(B) \times_{G(B)} G(A).$$

Definition. A morphism $F \to G$ in $\mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ is called *smooth* if for any *surjective* morphism $A \to B$ in Art_k, the map

$$F(A) \to F(B) \times_{G(B)} G(A)$$

is *surjective.*

Proposition 2.2. *Let $R \to S$ be a morphism in $\hat{\mathrm{Art}}_k$. Then $h_S \to h_R$ is smooth if and only if there exists an $n \in \mathbb{N}$ such that*

$$S \simeq R[[x_1, \ldots, x_n]].$$

Proof. For any $A \in \hat{\mathrm{Art}}_k$ the map $h_S(A) \to h_R(A)$ is induced by precomposition with $R \to S$. Suppose $h_S \to h_R$ is smooth and consider

$$t^*_{S/R} := \mathfrak{m}_S/(\mathfrak{m}_S^2 + \mathfrak{m}_R S).$$

The k-vector space $t^*_{S/R}$ is called the *Zariski cotangent space of S over R*, note that multiplication by k is induced from the action of $S/\mathfrak{m}_S \simeq k$. The Zariski cotangent space over the base field k will be refered to as just the *Zariski cotangent space.*

Set $x_1, \ldots, x_n \in S$ inducing a basis of $t^*_{S/R}$. And define $T = R[[X_1, \ldots, X_n]]$; we are going to show that $S \simeq T$.

There is a morphism of local R algebras

$$u_1 : S \to T/(\mathfrak{m}_T^2 + \mathfrak{m}_R T) : x_i \mapsto \overline{X}_i.$$

To show that this morphism is well defined, observe that $S/(\mathfrak{m}_S^2 + \mathfrak{m}_R S) \simeq k \oplus \mathfrak{m}_S/(\mathfrak{m}_S^2 + \mathfrak{m}_R S) = t^*_{S/R}$. So we may define an isomorphism between the two vector spaces $t^*_{S/R}$ and $t^*_{T/R}$ by mapping the basis x_i to $\overline{X}_i$. Adding on the vector space k to both sides and composing with the quotient map from S gives us u_1.

Now we apply smoothness of $h_S \to h_R$ to the surjection $T/\mathfrak{m}_T^2 \to T/(\mathfrak{m}_T^2 + \mathfrak{m}_R T)$.

Let g denote the corresponding sujective morphism:

$$\hom(S, T/\mathfrak{m}_T^2) \to \hom(S, T/(\mathfrak{m}_T^2 + \mathfrak{m}_R T) \times_{\hom(R, T/\mathfrak{m}_T^2 + \mathfrak{m}_R T)} \hom(R, T/\mathfrak{m}_T^2).$$

There is a natural morphism $R \overset{i}{\hookrightarrow} T/\mathfrak{m}_T^2$, simply as the inclusion of the coefficient ring. Via the smoothness hypothesis there exists a map

$$u_2 : S \to T/\mathfrak{m}_T^2$$

such that $g(u_2) = (u_1, i)$. In particular, u_2 lifts u_1.

By repeating this argument we may find

$$u_3 : S \to T/\mathfrak{m}_T^3$$

lifting u_2. Continuing in this manner and using the universal property of the inverse limit we get:

$$u : S \to T = \varprojlim (T/\mathfrak{m}_T^n).$$

By definition of u_1, the morphism u induces an isomorphism $t^*_{S/R} \to t^*_{T/R}$.

We wish to show that the induced map on the Zariski cotangent spaces, ie. the morphism $\mathfrak{m}_S/\mathfrak{m}_S^2 \to \mathfrak{m}_T/\mathfrak{m}_T^2$, is surjective. To that end note the exact sequence:

$$\begin{array}{ccccccccc} 0 & \to & \mathfrak{m}_R T/(\mathfrak{m}_T^2 \cap \mathfrak{m}_R T) & \to & \mathfrak{m}_T/\mathfrak{m}_T^2 & \to & t^*_{T/R} & \to & 0 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 0 & \to & \mathfrak{m}_R S/(\mathfrak{m}_S^2 \cap \mathfrak{m}_R S) & \to & \mathfrak{m}_S/\mathfrak{m}_S^2 & \to & t^*_{S/R} & \to & 0 \end{array}.$$

The rightmost vertical arrow is a surjection, therefore if we show that the leftmost vertical arrow is also a surjection it will follow that the middle arrow too is a surjection.

Notice that $\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_R T/(\mathfrak{m}_T^2 \cap \mathfrak{m}_R T)$ is an isomorphism. Since the map $u : S \to T$ is such that the inclusion $R \to T$ factors through as $R \to S \overset{u}{\to} T$, the isomorphism above factors through as:

$$\mathfrak{m}_R/\mathfrak{m}_R^2 \to \mathfrak{m}_R S/(\mathfrak{m}_S^2 \cap \mathfrak{m}_R S) \to \mathfrak{m}_R T/(\mathfrak{m}_T^2 \cap \mathfrak{m}_R T).$$

This implies that the leftmost vertical arrow has to be surjective.

If A is a ring and $\mathfrak{a} \subseteq A$ is an ideal then we define *the graded ring* of A with respect to $\mathfrak{a}$ as

$$G(A) := \bigoplus_{n=0}^{\infty} \mathfrak{a}^n/\mathfrak{a}^{n+1}.$$

Since the map between the cotangent spaces of S and T is surjective, we get a surjective map $G(S) \to G(T) = T$. Applying Lemma 10.23 of Atiyah-MacDonald (page 112) we conclude $u : S \to T$ is surjective.

Construct

$$v : T = R[[X_1, \ldots, X_n]] \to S$$

by picking $y_i \in u^{-1}(X_i)$ for each i and setting $v(X_i) = y_i$. (To show that such a map can be defined you may start with the polynomial ring over R and use the universal property of the localization at the origin followed by the universal property of completions.)

Since $u \circ v = \mathrm{Id}_T$, the morphism v must be injective. Notice also that v is local, R-linear and that it induces an isomorphism between the cotangent spaces. Using the graded rings again we may conclude that v is surjective. Hence, v is an isomorphism. $\square$

Exercise. We established the "only if" part of the proof. The converse is easy and is left as an exercise.

Definition 2.3. Let (R, ξ) be a pro-couple for $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ corresponding to a morphism $h_R \to F$. Then (R, ξ) is called a *hull* of F if the corresponding map $h_R \to F$ is smooth and the induced map

$$\hom(R, k[\varepsilon]) \to F(k[\varepsilon])$$

on tangent spaces is bijective.

We now aim to give sufficient conditions for a functor $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ to

i) have a hull

ii) be pro-representable.

To this end we introduce SCHLESSINGER'S CONDITIONS H_1, H_2, H_3 and H_4. Throughout the definitions, $f : A \to A'$ and $g : A'' \to A$ will be in Art_k. Such a couple induce a map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$$

which we will denote by $f * g$.

Property H_1. F is said to satisfy property H_1 if for any f and any *surjective* $g : A'' \to A$ yielding a principal small extension, the induced map $f * g$ is *surjective.*

Property H_2. F satisfies H_2 if we take $g : A[\varepsilon] \to A$ to be the quotient map and pick any $f : A \to A'$ then the induced map $f * g$ is bijective.

We will show later that if properties H_1 and H_2 hold then $F(k[\varepsilon])$ is canonically a k-vector space. So we may introduce the next property.

Property H_3. If F satisfies H_1, H_2 and if in addition $\dim_k F(k[\varepsilon]) < \infty$ then F is said to satisfy H_3.

Property H_4. If $g : A'' \to A$ is a principal small extension then for any f the induced map $f * g$ is bijective.

In the new few weeks we intend to prove the following theorem:

Schlessinger's Theorem 2.4. *Let $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (Sets))$. Then F has a hull if and only if F satisfies H_1, H_2 and H_3. Furthermore, F is pro-representable if and only if in addition F satisfies H_4.*

Next week. We will study the Schlessinger conditions on specific examples such as the Picard functor.

Deformation Theory

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Lecture 5

1 Recall

Last week we saw Schlessinger conditions $H_1, \dots, H_4$ on a functor $F : \mathrm{Art}_k \to$ (Sets) with $F(k)$ a point.

Schlessinger's theorem which we will start proving next week says that F is pro-representable when $H_1, \dots, H_4$ are satisfied.

The goal of this week is to get a feel for Schlessinger conditions. We are first going to prove that the Picard functor

$$\mathfrak{Pic}_M : \mathrm{Art}_k \to (\mathrm{Sets})$$

satisfies the Schlessinger conditions.

2 Picard Functor

Let X be a scheme over k and $M \in \mathrm{Pic}(X) \simeq H^1(X, \mathcal{O}_X^*)$. For $A \in \mathrm{Art}_k$ we have

$$\mathfrak{Pic}_M(A) := \{L \in \mathrm{Pic}(X_A) \mid L \otimes_A k = M\}.$$

Recall that $X_A := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(A)$.

For any morphism $A \to B$ in Art_k the induced morphism $X_B \to X_A$ give a pullback map

$$\mathrm{Pic}(X_A) \to \mathrm{Pic}(X_B).$$

For $L \in \mathrm{Pic}(X_A)$ we denote the pullback of L as $L \otimes_A B$. In the definition of $\mathfrak{Pic}_M(A)$ above k is the residue field $A \to A/\mathfrak{m}_A \simeq k$.

From now on we make the following two assumptions about X:

1. $H^0(X, \mathcal{O}_X) \simeq k$
2. $\dim_k H^1(X, \mathcal{O}_X) < \infty$.

For instance, if X is a proper algebraic variety then the assumptions are satisfied. We will first show that $\mathfrak{Pic}_M$ satisfies conditions H_1 and H_2. We will recall their definitions from Lecture 4.

F satisfies condition H_1 if for any $f : A' \to A$, $g : A'' \to A \in \mathrm{Art}_k$ with g a surjection giving a small principal extension the following map is surjective:

$$f * g : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

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F satisfies condition H_2 if $f * g$ is bijective when we take g to be the quotient map $g : k[\varepsilon] \to k$.

3 Preparation

Before we can prove that $\mathfrak{Pic}_M$ satisfies the first two Schlessinger conditions, we have to prepare two algebraic results.

Lemma 3.1. *Let A be a ring, $J \subseteq A$ a nilpotent ideal and $u : M \to N$ a morphism of A-modules such that N is flat over A.*

If $\overline{u} : M/J \to N/J$ is an isomorphism then so is $u : M \to N$.

Proof. Let Q be the cokernel of u and consider the exact sequence

$$M \xrightarrow{u} N \to Q \to 0.$$

Tensor this sequence of A-modules with A/J. Since tensoring is right exact we still have an exact sequence

$$M/JM \xrightarrow{\overline{u}} N/JN \to K/JK \to 0.$$

By assumption $\overline{u}$ is an isomorphism and thus $K/JK = 0$. Therefore $K = JK = J^n K$ for any $n \geq 0$. As J is nilpotent we conclude $K = 0$, ie. u is surjective.

Therefore we have a short exact sequence:

$$0 \longrightarrow H \longrightarrow M \xrightarrow{u} N \longrightarrow 0$$

Since N is flat over A, after tensoring with A/J the first derived term on the left is 0 and we get an exact sequence:

$$0 \longrightarrow H/JH \longrightarrow M/JM \xrightarrow{\overline{u}} N/JN \longrightarrow 0$$

Once again we conclude H/JH and thus H is 0. Therefore u is injective. $\square$

Exercise. Using this lemma prove that if M is flat over $A \in \mathrm{Art}_k$ then M is in fact free.

In setting up for the next result we need a definition. Note that we chose the geometric (ie. scheme theoretic) convention for our naming.

Definition 3.2. Let $B \to A$ be a ring morphism. If N is a B module then we will call $N \otimes_B A$ the *pullback of N to A*.

If M is an A-module isomorphic to the pullback of N, then there are B-module morphisms $u : N \to M$ (factoring through $N \to N \otimes_B A$) which induce an isomorphism $N \otimes_B A \to M$. We will call such morphisms *restriction maps.*

Lemma 3.3. *Suppose $A' \to A$ is a morphism and $A'' \to A$ is a surjective morphism with nilpotent kernel J.*

Let M be an A module, M' a free A' module and M'' a flat A'' module; such that M' and M'' pull back to M. Let $u' : M' \to M$ and $u'' : M'' \to M$ be restriction maps.

With $B = A' \times_A A''$ and $N = M' \times_M M''$ we may construct the following Cartesian diagram using the projection maps p' and p'':

$$\begin{tikzcd} N = M' \times_M M'' \arrow[r, "p''"] \arrow[d, "p'"] & M'' \arrow[d, "u''"] \\ M' \arrow[r, "u'"] & M \end{tikzcd}.$$

Under these circumstances, N is free over B. Moreover, N pulls back to M' and M'' while p' and p'' provide restriction maps.

Proof. Choose a basis $(x'_i)_{i\in I}$ of M' over A'. Since $u' : M' \to M$ is a restriction map, M is free over A with basis $(u'(x'_i))_{i\in I}$.

Since $M \simeq M''/JM''$ we may choose $x''_i \in M''$ such that $u''(x''_i) = u'(x'_i)$. The morphism

$$\bigoplus_{i\in I} A''x''_i \to M''$$

pulls back to an isomorphism on A. Since J is nilpotent we may apply Lemma 3.1 to conclude that M'' is free with basis $(x''_i)_{i\in I}$. As $u''(x''_i) = u'(x'_i)$ the pairs $(x''_i, x'_i) \in N$ form a free basis over B.

Exercise. Convince yourself of this fact.

Note also that the projection maps p' and p'' map the basis to the corresponding bases. Making them restriction maps. $\square$

Corollary 3.4. *Together with the hypotheses of Lemma 3.3, assume that L is a B module and that we have a commutative diagram of B modules:*

$$\begin{tikzcd} L \arrow[r, "q''"] \arrow[d, "q'"] & M'' \arrow[d, "u''"] \\ M' \arrow[r, "u'"] & M \end{tikzcd}.$$

By the universal property of fiber products we get a morphism $u : L \to N$ making the following diagram commutative:

$$\begin{tikzcd} L \arrow[dr, "u"] \arrow[drr, bend left, "q''"] \arrow[ddr, bend right, "q'"'] & & \\ & N \arrow[r, "p''"] \arrow[d, "p'"] & M'' \arrow[d, "u''"] \\ & M' \arrow[r, "u'"] & M \end{tikzcd}.$$

If q' is a restriction map then u is an isomorphism.

Proof. First, as an easy exercise, show that $B \to A'$ is surjective with nilpotent kernel J'. Use the explicit construction of product of rings.

Lemma 2 assures that the projection $p' : N \to M'$ is a restriction map. The commutativity of the diagram and our hypothesis on q' implies u must pull back to an isomorphism on A'. This pull back can be written in the form:

$$\overline{u} : L/J'L \to N/J'N.$$

Lemma 2 also says that N is free and thus flat over B. Since J' is nilpotent we may use Lemma 1 to conclude that u is itself an isomorphism. □

4 Main result

We are finally ready to prove the main result of this lecture.

Proposition 4.1. *Let $f : A' \to A$ and $g : A'' \to A$ be morphisms in Art_k with g a surjection. Then the natural map*

$$f * g : \mathfrak{Pic}_M(A' \times_A A'') \to \mathfrak{Pic}_M(A') \times_{\mathfrak{Pic}_M(A)} \mathfrak{Pic}_M(A'')$$

is a bijection. In particular $\mathfrak{Pic}_M$ satisfies H_1 and H_2.

Proof. For simplicity of notation let $\mathfrak{P}$ denote $\mathfrak{Pic}_M$. Let $(L', L'') \in \mathfrak{P}(A') \times_{\mathfrak{P}(A)} \mathfrak{P}(A'')$ with the common image $L \in \mathfrak{P}(A)$. Let B denote the product $A' \times_A A''$.

Let $|X|$ denote the underlying topological space of X. Note that the spectrum of an Artinian ring A consists of a point and so $|X_A| = |X|$.

We may thus consider the following commutative diagram of sheaves on $|X|$:

$$\begin{tikzcd} \mathcal{O}_{X_B} \arrow[r] \arrow[d] & \mathcal{O}_{X_{A''}} \arrow[d] \\ \mathcal{O}_{X_{A'}} \arrow[r] & \mathcal{O}_{X_A} \end{tikzcd}.$$

This induces a canonical morphism $\mathcal{O}_{X_B} \to \mathcal{O}_{X_{A'}} \times_{\mathcal{O}_{X_A}} \mathcal{O}_{X_{A''}}$ of sheaves on $|X|$.

Since A'' is Artinian, the kernel of $g : A'' \to A$ is nilpotent. Therefore we may apply Corollary 3.4 to conclude that the canonical map just defined is an isomorphism of sheaves of B modules:

$$\mathcal{O}_{X_B} \xrightarrow{\sim} \mathcal{O}_{X_{A'}} \times_{\mathcal{O}_{X_A}} \mathcal{O}_{X_{A''}}.$$

Therefore $N := L' \times_L L''$ is an invertible sheaf on X_B. By Lemma 3.3, N pulls back to L' and L'' on $X_{A'}$ and $X_{A''}$ respectively. In other words $f * g(N) = (L', L'')$ and thus $f * g$ is surjective.

It remains to verify injectivity of $f * g$. Let $M \in \mathfrak{P}(B)$ map to (L', L'') via $f * g$ and pull back to L over X_A. Notice that there is an automorphism $\theta : L \to L$ which makes the following diagram commute:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & L'' \arrow[dr, "q''"] & \\ L' \arrow[r, "q'"] & L \arrow[r, "\theta"] & L \end{tikzcd}.$$

We will now describe this automorphism θ. Recall that we assumed $k \to H^0(X, \mathcal{O}_X)$ is an isomorphism. This implies that the natural map

$$A \to H^0(X_A, \mathcal{O}_{X_A}) = H^0(X, \mathcal{O}_X \otimes_k A)$$

is an isomorphism. This is a consequence of the following exercise.

Exercise. The canonical map $k \to H^0(X, \mathcal{O}_X)$ is an isomorphism if and only if for all k-vector spaces V the canonical map $V \to H^0(X, \mathcal{O}_X \otimes_k V)$ is an isomorphism. [*Hint:* $\mathcal{O}_X \otimes_k V \simeq \bigoplus_{v \in \mathscr{B}} \mathcal{O}_X$, where $\mathscr{B}$ is a basis for V.]

Thus θ is multiplication by a unit a in A. Replacing q'' with $\frac{1}{a} \cdot q''$ we may assume θ is the identity map. That is $u'q' = u''q''$.

Now we may apply Corollary 3.4 to conclude that M is isomorphic to N. Proving that $f * g$ is injective. $\square$

Next week. We will show that $\mathfrak{Pic}_M$ satisfies H_3, H_4 and we will begin proving Schlessinger's Theorem.

Deformation Theory

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Lecture 6

The meaning of bonus exercises. While typing the notes, I (Emre) added bonus exercises to emphasize some non-trivial points in the argument that are not covered in lecture notes. They are occasionally very time consuming, so do them at your own risk. In anycase, they are by no means included in the exam, should you be interested in taking one. But, any run-of-the-mill exercises and anything that is said to be 'left to the reader' are from the lecture notes; ignore them at your own risk.

1 Finishing the proof from last week

Last week we showed that $\mathfrak{Pic}_M$ satisfies the first two Schlessinger conditions provided X is such that $H^0(X, \mathcal{O}_X) = k$ and $\dim_k H^1(X, \mathcal{O}_X) < \infty$. This week we will prove that $\mathfrak{Pic}_M$ further satisfies the last two Schlessinger conditions.

Remark. Condition H_4 requires that $f * g$ be bijective whenever g is a principle small extension. Previous week we showed that for any two morphisms f and g in Art_k, with g surjective, the induced map $f * g$ for $\mathfrak{Pic}_M$ is bijective. Hence $\mathfrak{Pic}_M$ satisfies H_4.

The fact that H_3 is satisfied by $\mathfrak{Pic}_M$ is a consequence of the following lemma.

Lemma 1.1. *If X is a scheme over k such that $H^1(X, \mathcal{O}_X)$ is finite dimensional, then $\mathfrak{Pic}_M(k[\varepsilon])$ is a finite dimensional k-vector space*

Proof.

Step 1:

We fixed M, a line bundle on X, to define $\mathfrak{Pic}_M$. Now we are going to show that we may assume $M = \mathcal{O}_X$ for this proof.

Let $L_0 \in \mathfrak{Pic}_M(k[\varepsilon])$ be the zero element of the vector space structure (in fact L_0 is the image of M with respect to the structure map $k \to k[\varepsilon]$). Define a map:

$$\begin{array}{rcl} \mathfrak{Pic}_M(k[\varepsilon]) & \to & \mathfrak{Pic}_{\mathcal{O}_X}(k[\varepsilon]) \\ L & \mapsto & L \otimes L_0^\vee. \end{array}$$

Since tensoring with a line bundle is an invertible operation, this map is a bijection.

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Bonus exercise. Show that the map above is a vector space isomorphism.

Step 2:

From this point onwards we may, and will, take $M = \mathcal{O}_X$. Let Y be $X_{k[\varepsilon]} = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon])$. Let $r : \mathrm{Spec}(k) \to \mathrm{Spec}(k[\varepsilon])$ be the map corresponding to the quotient map $k[\varepsilon] \to k$. Let $\pi : X \to \mathrm{Spec}(k)$ be the structure morphism. The inclusion of the central fiber is given by $\mu := \mathrm{Id}_X \times (r \circ \pi) : X \to Y$. By definition we have

$$\mathfrak{Pic}_{\mathcal{O}_X}(k[\varepsilon]) = \{L \in \mathrm{Pic}(Y) \mid \mu^* L \simeq \mathcal{O}_X\} = \ker(H^1(Y, \mathcal{O}_Y^*) \xrightarrow{\mu^*} H^1(X, \mathcal{O}_X^*)).$$

Step 3:

Now we will concentrate on the map between the cohomologies. First, recall that μ is a homeomorphism of the underlying topological spaces of X and Y. The morphism between the structure sheaves $\mu^\# : \mathcal{O}_Y \to \mathcal{O}_X$ is simply the projection on to the first factor where we make the obvious identification $\mathcal{O}_Y = \mathcal{O}_X \oplus \varepsilon \mathcal{O}_X$.

Further note that we have $\mathcal{O}_Y^* = \mathcal{O}_X^* \cdot (1 + \varepsilon \mathcal{O}_X)$. This gives an exact sequence:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_Y^* \xrightarrow{\mu^\#} \mathcal{O}_X^* \longrightarrow 0$$

where the exponential map is defined so that $\exp(f) = 1 + \varepsilon f$. Note that the surjection indeed coincides with the projection onto the first factor. We may take the associated long exact sequence, observing that $\mu^\#$ obviously surjects on the level of global sections, we are left with:

$$0 \longrightarrow H^1(\mathcal{O}_X) \longrightarrow H^1(\mathcal{O}_Y^*) \xrightarrow{\mu^*} H^1(\mathcal{O}_X^*) \ .$$

Consequently,

$$\mathfrak{Pic}_{\mathcal{O}_X}(k[\varepsilon]) \simeq H^1(\mathcal{O}_X).$$

But by hypothesis the latter vector space is finite dimensional over k and thus so is $\mathfrak{Pic}_{\mathcal{O}_X}(k[\varepsilon])$ once you prove the following.

Bonus exercise. The map $\mathfrak{Pic}_{\mathcal{O}_X}(k[\varepsilon]) \simeq H^1(\mathcal{O}_X)$ is a k-vector space isomorphism.

□

Summing up the results so far we have:

Theorem 1.2. *Let X be a scheme over k such that $H^0(X, \mathcal{O}_X) = k$ and $\dim_k H^1(X, \mathcal{O}_X) < \infty$. For any $M \in \mathrm{Pic}(X)$ the functor $\mathfrak{Pic}_M$ satisfies the four Schlessinger's conditions H_1, H_2, H_3 and H_4.*

2 Schlessinger's theorem

Recall that a natural transformation $\nu : F \to G$ in $\mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ is called smooth if for any surjection $f : A \to B$ the induced map

$$f^\nu : F(A) \to F(B) \times_{G(B)} G(A)$$

is surjective.

Recall also that we defined the 'tangent space' of F to be $t_F := F(k[\varepsilon])$ which in certain cases will have a k-vector space structure. As for a ring R we will refer to t_{h_R} as t_R.

Finally, recall that a *hull* of F is a couple (R, r) with $R \in \hat{\mathrm{A}}\mathrm{rt}_k$ and $r \in \hat{F}(R)$ such that the induced map $\tilde{r} : h_R \to F$ is smooth and the map between the tangent spaces $t_R \to t_F$ is an isomorphism.

We will now prove:

Theorem 2.1. *Assume that $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (\mathit{Sets}))$ has a hull. Then F satisfies the first three Schlessinger conditions.*

As always let $f : A' \to A$ and $g : A'' \to A$ be in Art_k. Let (R, r) be a hull of F.

To be able to distinguish the two induced maps, we use the following notation:

$$\begin{aligned} f *_F g &\;:\; F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \\ f *_{h_R} g &\;:\; h_R(A' \times_A A'') \to h_R(A') \times_{h_R(A)} h_R(A''). \end{aligned}$$

Proof of condition 1. We will show that smoothness of $\tilde{r} : h_R \to F$ implies that $f *_F g$ is surjective, for any f and g as above. By elementary diagram chasing we construct a commutative diagram:

$$\begin{tikzcd} F(A' \times_A A'') \arrow[r, "f *_F g"] & F(A') \times_{F(A)} F(A'') \\ h_R(A' \times_A A'') \arrow[u, two heads] \arrow[r, "f *_{h_R} g"] & h_R(A') \times_{h_R(A)} h_R(A'') \arrow[u, two heads] \end{tikzcd}$$

Check that the smoothness condition implies, for any $B \in \mathrm{Art}_k$, that $h_R(B) \to F(B)$ is surjective. (Hint: Take the map to be $B \to k$.) Therefore, the vertical arrows are surjective by smoothness.

By showing that the bottom horizontal map is surjective we may conclude that the top horizontal map is surjective. Since h_R is a hom functor, it preserves fiber products. Therefore, the bottom horizontal arrow is a bijection. $\square$

Proof of condition 2. We take g to be the map $k[\varepsilon] \to k$. We wish to prove that $f *_F g$ is a *bijection* for any f. Denote $A' \times_k k[\varepsilon]$ by $A'[\varepsilon]$. Using also the tangent space notation, the commutative diagram in the previous proof becomes:

$$\begin{tikzcd} F(A'[\varepsilon]) \arrow[r, "f *_F g"] & F(A') \times t_F \\ h_R(A'[\varepsilon]) \arrow[u, two heads] \arrow[r, "\sim"] & h_R(A') \times t_R \arrow[u, two heads] \end{tikzcd}$$

We will show that $f *_F g$ is injective. Take $x, y \in F(A'[\varepsilon])$ which map to the same $(\alpha, \beta) \in F(A') \times t_F$. Fix a lift $a \in h_R(A')$ of $\alpha \in F(A')$. By smoothness the following map is surjective:

$$h_R(A'[\varepsilon]) \to h_R(A') \times_{F(A')} F(A'[\varepsilon]).$$

Therefore we may pick lifts $\varphi_x, \varphi_y \in h_r(A'[\varepsilon])$ of (a, x) and (a, y) respectively. Using the bottom horizontal arrow of our commutative diagram, we may associate these lifts to pairs (a, φ'_x) and (a, φ'_y) in $h_R(A') \times t_R$, uniquely. Since (R, r) is a hull, the rightmost vertical arrow is a bijection between the second components. But both φ'_x and φ'_y map to the same element β so they must be equal.

It follows that x and y are equal. This proves injectivity of $f *_F g$ and having established surjectivity in the previous proof, we are done. □

Proof of condition 3. This is immediate because $t_F \simeq t_R$ by hypothesis. But $t_R \simeq R/\mathfrak{m}_R \in \mathrm{Art}_k$ which implies that $\dim_k t_R < \infty$. □

3 Additional Lemmas

For future reference we will prove two lemmas. This first implies a fact we have been using so far: If F satisfies H_2 then t_F has a canonical vector space structure. We will be interested in the following kind of functors.

Definition 3.1. A functor satisfying the first two Schlessinger conditions is said to be a *functor with good deformation theory.*

Let us first introduce a notation. For a k-vector space V we define $k[V]$ to be the k-algebra where elements in V give a square zero ideal. Note for instance that $k[V] \times_k k[W] \simeq k[V \oplus W]$.

Bonus exercise. Prove that if F satisfies H_2 then it satisfies the hypothesis of the following lemma.

Lemma 3.2. *Suppose F is a functor such that for any two finite dimensional k-vector spaces V and W the canonical map $F(k[V] \times_k k[W]) \to F(k[V]) \times F(k[W])$ is a bijection. Then $F(k[V])$, and in particular $t_F = F(k[\varepsilon])$, has a canonical vector space structure. Moreover, there is a natural isomorphism:*

$$t_F \otimes V \to F(k[V]).$$

Proof. The addition on V defines an "addition" map on $k[V]$ as follows:

$$\begin{array}{ccc} k[V] \times_k k[V] & \to & k[V] \\ (c+v, c+w) & \to & (c, v+w) \end{array}$$

where $c \in k$ and $v, w \in V$. Similarly, scalar multiplication on V defines scalar multiplication on $k[V]$, e.g. $a \in k$ takes (c, v) to (c, av). Using the canonical isomorphism (induced by the projection maps)

$$F(k[V] \times_k F[V]) \to F(k[V]) \times F(k[V])$$

we can carry the addition and scalar multiplication maps on $k[V]$ to the set $F(k[V])$ via F. Its easy to check that these satisfy the necessary vector space axioms making $F(k[V])$ a vector space. (e.g. by showing that F preserves the commutativity of necessary diagrams.)

Since $hom_C(k[\varepsilon], k[V]) \simeq V$, which is a k-linear map with respect to the natural vector space structures on both sides, given $(x, v) \in t_F$ we can treat $v \in V$ as a map $k[\varepsilon] \to k[V]$ thereby define $F(v)(x) \in F(k[V])$. This is a k-linear map and it induces the map

$$t_F \otimes V \to F(k[V]).$$

Notice $k[V] \simeq \prod_{i=1}^{\dim V} k[\varepsilon]$ where the product is taken over k. Thus it is easy to see that the map above is an isomorphism. $\square$

We will need the following exercise for the final lemma:

Exercise. F satisfies H_1 iff for any $f : A' \to A$ and any *small extension* $g : A'' \to A$ in Art_k the map $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ is surjective. (That is, the condition of being principle may be omitted.)

Lemma 3.3. *Suppose F is a functor with good deformation theory. Let $B \to A$ be a small extension in Art_k with kernel M. Then there is a canonical transitive action of $F(k[M]) = t_F \otimes_k M$ on the fibers of $F(B) \to F(A)$.*

Proof. First off let $V := t_F \otimes_k M$.

To see why we should expect such an action, and motivate the remaining part of the proof, we will first assume $B \to A$ is split. That is $B \simeq A[M]$. By the previous lemma and using H_2 we have

$$F(B) \simeq F(A) \times F(k[M]) \simeq F(A) \times V$$

In this case the fiber of the map $F(B) \to F(A)$ is just V and the action of V on the fibers is clearly transitive as well as canonical (because the bijections above are canonical).

As for the general proof, consider $B \times_A B$. It is left as an exercise to show that $B[M] \simeq B \times_A B$, where the isomorphism is canonical with respect to their universal properties (Hint: The map is defined by $(b, m) \to (b, b + m)$).

Using H_2 again we conclude $F(B[M]) = F(B) \times V$. This gives us the following sequence of maps:

$$F(B) \times V \simeq F(B[M]) \simeq F(B \times_A B) \twoheadrightarrow F(B) \times_{F(A)} F(B)$$

where the last arrow is a surjection due to H_1. The projection on to the first factor gives us the first factor we began with and projection on to the second factor allows us to define the V action on the set $F(B)$. The fibers of $F(B) \to F(A)$ are obviously preserved, simply due to the codomain of the map above. Transitivity of the action follows from the surjectivity of that map.

Bonus exercise. Check that this is indeed a *group action*. (Hint: Express this condition using diagrams or you are in trouble.)

$\square$

Deformation Theory

Michael Kemeny*

Lecture 7

1 Schlessinger's theorem, the other direction

Now we wish to prove that the first three Schlessinger's conditions imply the existence of a hull. This is the converse of what we proved last week.

Theorem 1.1. *Let $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (Sets))$ satisfy H_1, H_2, H_3. Then F admists a hull.*

Proof. Our goal is to construct $R \in \hat{\mathrm{Art}}_k$ and an element $u \in \hat{F}(R)$ such that the map $h_R \to F$ induced by u is smooth and is an isomorphism on the tangent spaces. By definition of $\hat{\mathrm{Art}}_k$, $R \simeq \varprojlim R/\mathfrak{m}_R^n$. Therefore, in defining R we may as well start with a sequence of small surjections and take their limit:

$$\dots R_{i+1} \twoheadrightarrow R_i \twoheadrightarrow R_{i-1} \twoheadrightarrow \dots \twoheadrightarrow R_1 \twoheadrightarrow R_0 = k \in \mathrm{Art}_k\,.$$

In constructing the element u we need to choose a sequence of elements $u_i \in F(R_i)$ that are compatible with respect to the maps above.

Step 1

Observe that for any $R \in \hat{\mathrm{Art}}_k$ we have a surjection $R \twoheadrightarrow R/\mathfrak{m}^2 \simeq k[M]$ where $M = \mathfrak{m}_R/\mathfrak{m}_R^2$. This gives an isomorphism $t_R \to t_{k[M]}$. Hence the discussion below describes how we should pick M and thus R_1 as well as u_1 in order to ensure that $t_R \to t_F$ is an isomorphism.

Let $R_1 = k[M]$ be the square zero extension for some k-vector space M. Note that $t_{R_1} = \hom_{\mathrm{Art}_k}(k[M], k[\varepsilon])$. We have a natural pairing:

$$\begin{array}{rcl} t_{R_1} \times F(k[M]) & \to & F(k[\varepsilon]) = t_F \\ (f,x) & \mapsto & F(f)(x). \end{array}$$

Recall that we have an isomorphism

$$\begin{array}{rcl} M & \to & \hom_{\mathrm{Art}_k}(k[\varepsilon], k[M]) \\ m & \mapsto & (\varepsilon \mapsto m), \end{array}$$

which allowed us to define the isomorphism

$$\begin{array}{rcl} t_F \otimes M & \to & F(k[M]) \\ v \otimes m & \mapsto & F(\varepsilon \mapsto m)(v). \end{array}$$

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Similarly we have an isomorphism

$$\begin{array}{rcl} M^\vee & \to & \hom_{\mathrm{Art}_k}(k[M], k[\varepsilon]) \\ \varphi & \mapsto & (s + \varphi \cdot \varepsilon) \end{array}$$

where $s : k[M] \to k$ is the canonical projection.

Using these identifications the pairing above becomes

$$\begin{array}{rcl} M^\vee \times (t_F \otimes M) & \to & t_F \\ (\varphi, v \otimes m) & \mapsto & \varphi(m) \cdot v. \end{array}$$

If we choose an element $u_1 = \sum_i v_i \otimes m_i \in t_F \otimes M$ such that the v_i form a basis of t_F and the m_i form a basis of M then it is easily checked that the pairing above gives an isomorphism:

$$t_{R_1} = M^\vee \to t_F.$$

Step 2

Let $S = k[[T_1, \ldots, T_r]]$ where $r = \dim_k t_F$ and T_i's form a basis for $t_F^\vee$. Further let $\eta = (T_1, \ldots, T_r)$ be the maximal ideal. We let $R_0 = S/\eta = k$ and $R_1 = S/\eta^2 = k[t_F^\vee]$. We pick the only element in $F(R_0)$ to be u_0 and we choose $u_1 \in F(R_1)$ such that $t_{R_1} \to t_F$ is an isomorphism. (In fact, for fun, we can choose u_1 so that $t_{R_1} = (t_F^\vee)^\vee \to t_F$ is the canonical isomorphism.)

We will construct the other pairs (R_n, u_n) by induction as follows. We want $R_i = S/J_i$ for some ideal $\eta J_{i-1} \subseteq J_i \subseteq J_{i-1}$ and we want $F(R_i \twoheadrightarrow R_{i-1})(u_i) = u_{i-1}$. Suppose (R_i, u_i) has been constructed in this way for $0 \leq i \leq n-1$. Let $\mathcal{I}_n$ denote the set of all ideals $\eta J_{n-1} \subseteq J \subseteq J_{n-1}$ such that there exists $u' \in F(S/J)$ satisfying $F(S/J \twoheadrightarrow R_{i-1})(u') = u_{i-1}$.

We claim that $\mathcal{I}_n$ is closed under arbitrary intersections. Therefore we may define J_n to be the smallest element, or the intersection of all elements, in $\mathcal{I}_n$. Leaving the proof of this claim to the next step, let us continue in constructing R.

Having defined J_n and thus $R_n = S/J_n$ we pick any $u_n \in F(R_n)$ lifting u_{n-1}. So that we may define

$$R = \varprojlim R_n, \quad u = \varprojlim u_n \in \hat{F}(R).$$

Remark. In general $R/\mathfrak{m}_R^n \not\simeq R_n$. However the fact that $R \simeq S/\cap_i J_i$ means $\mathfrak{m}_R^n \subseteq J_n/\cap_i J_i$ for all n. This is why we can define u as the limit of u_n's. We won't go into this here.

It is clear that $t_R = h_R(k[\varepsilon]) = h_{R_1}(k[\varepsilon]) = t_{R_1}$ and the map $t_R \to t_F$ induced by u coincides with the map $t_{R_1} \to t_F$ induced by u_1. That is, $h_R \to F$ induced by u gives an isomorphism of the tangent spaces. It remains to show that this map is smooth, which we postpone to Step 4 of this proof.

Step 3

We are going to show that $\mathcal{I}_n$ defined above is closed under arbitrary intersections. Notice however that the ideals in $\mathcal{I}_n$ give vector subspaces of the finite dimensional vector space $J_{n-1}/\eta J_{n-1}$. Conversely we claim that any vector space V in $J_{n-1}/\eta J_{n-1}$ pulls back to an ideal containing ηJ_{n-1} and contained in J_{n-1}. The pull back can be performed via the ring quotient $S \to S/\eta J_{n-1}$ hence if $V \subset S/\eta J_{n-1}$ is an ideal, its pull back will be an ideal. The exercise below provides the missing ingredient.

Bonus exercise. Let $V \subset J_{n-1}/\eta J_{n-1}$ be a vector space. Show that the multiplication action of $S/\eta J_{n-1}$ on V factors through $S/\eta = k$. Hence V is a square zero ideal in $S/\eta J_{n-1}$.

An intersection of an arbitrary collection of subspaces within a finite dimensional vector space can be realized as the intersection of a finite subcollection of these subspaces. Hence, it is enough to show that $\mathcal{I}_n$ is closed under finite intersections, or by induction, under the intersection of any two ideals.

Pick $I, J \in \mathcal{I}_n$ and let $K = I \cap J$. By the following exercise we will assume that $I + J = J_{n-1}$.

Exercise. Given $I, J \in \mathcal{I}_n$ we may expand J such that $I + J = J_{n-1}$ without leaving $\mathcal{I}_n$ (Easy, but check.) Thus we have

$$S/(I \cap J) \simeq S/I \times_{R_{n-1}} S/J.$$

Using the exercise we get a natural map

$$F(S/K) \simeq F(S/I \times_{R_{n-1}} S/J) \to F(S/I) \times_{F(R_{n-1})} F(S/J).$$

Moreover this map is surjective by H_1 because $S/J \to S/J_{n-1}$ is a small extension (as $\eta J_{n-1} \subset J$). By definition of $\mathcal{I}_n$ we have $u_I \in F(S/I)$ and $u_J \in F(S/J)$ both of which map to $u_{n-1} \in F(R_{n-1})$. Hence $(u_I, u_J) \in F(S/I) \times_{F(R_{n-1})} F(S/J)$. Consequently there is a $u_K \in F(S/K)$ mapping to (u_I, u_J) and thus to u_{n-1}. That is, $K \in \mathcal{I}_n$.

Step 4

Now we will prove that the map $\nu : h_R \to F$ just constructed is smooth. That is, for any $\varphi : B \twoheadrightarrow A$ in Art_k we need to show that

$$\nu^\varphi : h_R(B) \to h_R(A) \times_{F(A)} F(B)$$

is surjective. Let us simplify the problem using the following lemma.

Lemma 1.2. *A natural transformation $\nu : G \to F$ in $\mathrm{Fun}^*(\mathrm{Art}_k, (Sets))$ is smooth iff for any* principal small extension *$\varphi : B \twoheadrightarrow A$ in Art_k the map*

$$\nu^\varphi : h_R(B) \to h_R(A) \times_{F(A)} F(B)$$

is surjective.

Proof of lemma. Since $\mathfrak{m}_B^n = 0$ for some n, we can factor the surjection φ as follows. Let $I = \ker \varphi$ and $A_i = B/\mathfrak{m}_B^i I$ so that we have

$$B = A_n \to A_{n-1} \to \cdots \to A_1 = A$$

where each map is a small extension. Similarly we can factor any small extension by principal small extensions.

Let us outline the induction step. Suppose $A_3 \overset{\varphi_2}{\to} A_2 \overset{\varphi_1}{\to} A_1$ are two maps such that ν^{φ_1} and ν^{φ_2} are surjective. We will show that $\nu^{\varphi_1 \circ \varphi_2}$ is surjective. Consider the diagram below:

$$\begin{tikzcd}
F(A_3) \arrow[r] \arrow[d] & F(A_2) \arrow[r] \arrow[d] & F(A_1) \arrow[d] \\
G(A_3) \arrow[r] & G(A_2) \arrow[r] & G(A_1)
\end{tikzcd}$$

Pick $g_3 \in G(A_3)$ and $f_1 \in F(A_1)$ lying over $g_1 \in G(A_1)$. Let $g_2 \in G(A_2)$ be the image of g_3. By surjectivity of ν^{φ_1} there is an element $f_2 \in F(A_2)$ which maps to g_2 and f_1. By surjectivity of ν^{φ_2} there is an element $f_3 \in F(A_3)$ which maps to g_3 and f_2. This proves surjectivity of $\nu^{\varphi_1 \circ \varphi_2}$. □

In light of the lemma above, we take a principal small extension $\varphi : B \to A$ to test surjectivity of ν^{φ}. Fix $b \in F(B)$ with image $a \in F(A)$. If $\alpha \in h_R(A)$ maps to a, that is $\nu_A(\alpha) = F(\alpha)(u) = a$, then we need to check if (α, b) lifts to $h_R(B)$. In other words, we need to find a map $\beta : R \to B$ such that $F(\beta)(u) = b$ and α factors through β. First we will show that we need not worry about b and that lifting the map α is enough.

Suppose we find some β lifting α, that is $\alpha = \varphi \circ \beta$. Then $F(\beta)(u) = b'$ where b' maps to a. Then we can modify β so as to hit b. Indeed, $F(B) \to F(A)$ has fibers which admit transitive $t_F \otimes \ker \varphi$ action on them. Note also that since the extension φ is principal $t_F \otimes \ker \varphi \simeq t_F$. So there is a $x \in t_F$ whose action on $F(B)$ carries b' to b. Recall in constructing this action we used the following sequence of maps

$$B \times_k k[I] \simeq B \times_A B \overset{\pi_2}{\to} B.$$

Pick $y \in t_R = h_R(k[I])$ which maps to x. Then the map $R \overset{(\beta,y)}{\to} B \times_k k[I]$ composed with the sequence of maps above yields a map to B which takes u to b.

Now we focus on finding a lift of α. Choose the smallest n such that α factors through $R_n \to A$. Since the extension $B \to A$ is small any lift β of α will factor through as $R_{n+1} \to B$. So we are looking for a map v to complete the following diagram:

$$\begin{tikzcd}
R_{n+1} \arrow[r, dashed, "v"] \arrow[d] & B \arrow[d] \\
R_n \arrow[r] & A
\end{tikzcd}$$

In order to use the minimality of the defining ideal of R_{n+1} we need to bring S into the picture. So observe that S being a power series ring, we can define a lift w completing the diagram:

$$\begin{tikzcd}
 & & B \arrow[d] \\
S \arrow[r] \arrow[urr, dashed, "w"] & R_n \arrow[r] & A
\end{tikzcd}$$

Therefore, the existence of v is equivalent to completing the following diagram

$$\begin{tikzcd}
S \arrow[r, "w"] \arrow[d] & R_n \times_A B \arrow[d, "\pi_1"] \\
R_{n+1} \arrow[r] \arrow[ur, dashed, "v"] & R_n
\end{tikzcd}$$

If π_1 has a section then obviously v exists. Otherwise we use the following lemma to conclude that w is surjective:

Lemma 1.3. *If a principal small extension $\varphi : B \to A$ doesn't have a section it is* essential. *That is, if a map $w : S \to B$ surjects onto A after composing with φ then w itself is surjective.*

The proof of this lemma is left for the following week. Let $J = \ker w$. It is clear that $\eta J_n \subseteq J \subseteq J_n$ and if we further show that u_n lifts to $S/J = R_n \times_A B$ then we may conclude $J \in \mathcal{I}_{n+1}$ from which it follows by minimality of $J_{n+1} \in \mathcal{I}_{n+1}$ that v exists.

To lift u_n observe that H_1 implies the map below is surjective

$$F(S/J) = F(R_n \times_A B) \to F(R_n) \times_{F(A)} F(B).$$

The element $(u_n, b)_a$ belongs to the right hand side. Therefore, there is an element $u' \in F(S/J)$ mapping to u_n. This concludes the proof that $h_R \to F$ is smooth. $\square$

Deformation Theory

Michael Kemeny*

Lecture 8

1 Theorem on pro-representability

We are now going to prove the final piece of Schlessinger's theorem.

Theorem 1.1. *Let $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (Sets))$ then F is pro-representable iff F satisfies H_1, H_2, H_3, H_4.*

Proof.
($\Longrightarrow$) Suppose F is pro-representable, i.e. there is $\nu : h_R \xrightarrow{\sim} F$. Clearly this map is also a hull of F therefore the conditions H_1, H_2, H_3 are satisfied. On the other hand, condition H_4 is immediate for any hom functor (by the universal property of fiber products).
($\Longleftarrow$) Conditions H_1, H_2, H_3 imply that there is a hull $\nu : h_R \to F$. We are going to prove that condition H_4 implies that ν is an isomorphism. What must be checked is that for any $A \in \mathrm{Art}_k$ the map $\nu_A : h_R(A) \to F(A)$ is a bijection. We are going to prove this by induction on the length of A.

Definition 1.2. Let $A \in \mathrm{Art}_k$. By *length of A*, denoted $\operatorname{len} A$, we will mean the largest integer $n \geq 1$ such that there is a sequence of ideals

$$0 \subsetneq I_1 \subsetneq \cdots \subsetneq I_n = A.$$

If $\operatorname{len} A = 1$ then $A = k$ and obviously ν_k is a bijection (between two sets of size one). If $\operatorname{len} A' = n \geq 2$ then take a principal small extension

$$0 \to I \to A' \to A \to 0.$$

Necessarily $\operatorname{len} A = n - 1$ and by induction hypothesis we may assume that ν_A is bijective. Combining this result with smoothness we get

$$h_R(A') \twoheadrightarrow h_R(A) \times_{F(A)} F(A') \simeq F(A').$$

That is, $\nu_{A'}$ is surjective. We now would like to show that $\nu_{A'}$ is injective. Consider the following diagram:

$$\begin{array}{ccc} h_R(A') & \longrightarrow & F(A') \\ \downarrow & & \downarrow \\ h_R(A) & \xrightarrow{\sim} & F(A) \end{array}$$

*TeX by Emre Sertöz. Please email suggestions or corrections to `emresertoz@gmail.com`

Recall that on $h_R(A')$ we have a $I \otimes t_R$ action that is transitive on the fibers of $h_R(A' \to A)$. Similarly we have a $I \otimes t_F$ action on $F(A')$ that is transitive on the fibers of $F(A' \to A)$.

Bonus exercise. Identifying the additive groups $I \otimes t_R$ and $I \otimes t_F$ with the isomorphism $\nu_{k[\varepsilon]}$ show that the diagram above is equivariant under these actions.

Exercise. Condition H_4 implies that the action of $I \otimes t_F$ on the fibers of $F(A' \to A)$ is in fact *free.*

Now suppose $x, y \in h_R(A')$ map to the same element $z \in F(A')$. Due to the bijectivity of the bottom horizontal arrow, x and y lie in the same fiber of $h_R(A' \to A)$. Hence there is an element $v \in I \otimes t_F$ such that $v \cdot x = y$. Using the equivariance of the map we conclude $v \cdot z = z$. The group action on each fiber of $F(A' \to A)$ is free by the exercise above hence $v = 0$. Therefore, $x = y$. $\square$

2 Tangent-obstruction theories

2.1 Definitions

A functor $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ is said to admit a tangent-obstruction theory if there exists finite dimensional k-vector spaces T_1 and T_2 satisfying the following three conditions.

Condition 1

For every small extension $0 \to M \to B \xrightarrow{\pi} A \to 0$ there exists an "exact sequence" of sets:

$$T_1 \otimes_k M \rightsquigarrow F(B) \xrightarrow{F(\pi)} F(A) \xrightarrow{\mathrm{ob}} T_2 \otimes_k M$$

Squiggly arrow denotes a $T_1 \otimes_k M$ action on the set $F(B)$ and exactness means the following are satisfies:

1. $\mathrm{ob}^{-1}(0) = \mathrm{Im}(F(\pi))$ (*Exactness at* $F(A)$).
2. The map $F(\pi)$ is invariant under the action of $T_1 \otimes M$. Furthermore, the action of $T_1 \otimes M$ on each fiber is transitive. (*Exactness at* $F(B)$).

Condition 2

If $A = k$ then the action of $T_1 \otimes M$ on $F(B)$ is free. Note that, $A = k$ implies $B \simeq k[M]$. Using the proof of Proposition 2.3 we get a canonical isomorphism of *vector spaces*:

$$F(k[M]) \simeq T_1 \otimes M.$$

Condition 3

The association of an exact sequence of sets to a small extension given in Condition 1 is functorial. To describe what this means we need to define what morphisms are between these objects. A morphism of small extensions

$\varphi : e_1 \to e_2$ is simply a triplet $(\varphi_M, \varphi_B, \varphi_A)$ making the following diagram commutative:

$$\begin{array}{ccccccccccc}
e_1 : & & 0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
& & & & \big\downarrow{\scriptstyle \varphi_M} & & \big\downarrow{\scriptstyle \varphi_B} & & \big\downarrow{\scriptstyle \varphi_A} & & \\
e_2 : & & 0 & \longrightarrow & M' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

In general, an exact sequence of sets will consist of the following data $G \rightsquigarrow S_1 \to S_2 \to H$ where G and H are groups, S_1 and S_2 are sets. The conditions of exactness are exactly as before. Hence a morphism of two exact sequences of sets $\psi : f_1 \to f_2$ is given by four morphisms $(\psi_G, \psi_1, \psi_2, \psi_H)$ such that the following diagram "commutes":

$$\begin{array}{ccccccccc}
f_1 : & & G & \rightsquigarrow & S_1 & \longrightarrow & S_2 & \longrightarrow & H \\
& & \big\downarrow{\scriptstyle \psi_G} & & \big\downarrow{\scriptstyle \psi_1} & & \big\downarrow{\scriptstyle \psi_2} & & \big\downarrow{\scriptstyle \psi_H} \\
f_2 : & & G' & \rightsquigarrow & S_1' & \longrightarrow & S_2' & \longrightarrow & H'
\end{array}$$

Here, by commutativity we mean that the map ψ_1 is equivariant with respect to ψ_G and for the rest of the diagram the notion of commutativity applies as usual.

Condition 1 assigns to a morphism $\varphi : e_1 \to e_2$ the following morphism $(\mathrm{Id} \otimes \varphi_M, F(\varphi_B), F(\varphi_A), \mathrm{Id} \otimes \varphi_M)$. Here we require this assignment to respect the composition of morphisms.

2.2 Main theorem we are aiming for

Our goal for the next few weeks is to prove the following theorem.

Theorem 2.1. *Let $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (Sets))$ be pro-represented by $R \in \hat{Art}_k$. Then the following hold.*

- *F admits a tangent-obstruction theory with $T_1 = t_F = t_R$.*
- *Let $d = \dim t_F$. Recall that $R \simeq k[[x_1, \ldots, x_d]]/J$ for some $J \subseteq \eta^2$ where $\eta = (x_1, \ldots, x_n)$. We may take $T_2 = J/\eta J$.*

First note that if J does not lie in η^2 then $\dim t_R < d$ which can not happen. Observe also that finding a tangent-obstruction theory for a pro-representable functor F reveals quite a bit about $R \in \hat{\mathrm{Art}}_k$ representing F. Nevertheless, we said that "we may take" $T_2 = J/\nu J$ because, in general, the obstruction space T_2 is not unique for F.

Exercise. Suppose F admits a tangent-obstruction theory (T_1, T_2). Show that for any k-vector space T_2' containing T_2 we may define a tangent-obstruction theory for F using the pair (T_1, T_2').

For our next result we need the following lemma.

Lemma 2.2. *If F admits a tangent-obstruction theory then for any k-vector space M the set $F(k[M])$ admits a canonical k-vector space structure. In particular t_F is a vector space.*

Proof. We are going to use Lemma 3.2 of Lecture 6 which states that our claim holds if for any two k-vector spaces M_1 and M_2 the canonical map below is a bijection:

$$F(k[M_1] \times_k k[M_2]) \to F(k[M_1]) \times F(k[M_2]).$$

Let $U = F(k[M_1] \times_k k[M_2])$ and $V_i = F(k[M_i])$ for $i = 1, 2$. By abuse of notation denote $F(\pi_i) : U \to V_i$ also by π_i. Choose a tangent-obstruction theory (T_1, T_2) and use the following morphism of the exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & M_1 \oplus M_2 \arrow[r] \arrow[d] & k[M_1] \times_k k[M_2] \arrow[r] \arrow[d, "\pi_i"] & k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_i \arrow[r] & k[M_i] \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

Using Conditions 1 and 3 we get a functorial assignment to these morphisms. Moreover, Condition 2 guarantees that the resulting actions are free. Fix the image of $F(k)$ in U and in V_i's to get the following diagram where the horizontal maps are a result of acting on these distinguished points and are bijective:

$$\begin{tikzcd}
(M_1 \oplus M_2) \otimes T_1 \arrow[r] \arrow[d] & U \arrow[d] \\
M_i \otimes T_1 \arrow[r] & V_i
\end{tikzcd}$$

Since $M_1 \oplus M_2 \simeq M_1 \times M_2$ we also get $U \simeq V_1 \times V_2$ via the canonical maps, which was to be proven. ☐

Proposition 2.3. *Let $F \in \mathrm{Fun}^*(\mathrm{Art}_k, (Sets))$ be a functor admitting a tangent-obstruction theory (T_1, T_2). Then there is a canonical isomorphism $T_1 \simeq t_F$.*

Proof. In fact we can prove a little more with no extra work. For any k-vector space M there is a canonical isomorphism $M \otimes T_1 \simeq F(k[M])$, where the latter has a vector space structure by the previous lemma.

We fix the point $x_0 \in F(k[M])$ which is the image of $F(k)$. This establishes a bijection $f : M \otimes T_1 \to F(M)$ as in the proof of the previous lemma. The point here is to show that this bijection is in fact an isomorphism of vector spaces. We will show that the map is additive leaving the (easier) proof that it respects scaling by k to the reader.

Let $V = F(M)$ and $W = M \otimes T_1$. We wish to show that the following diagram commutes, where the vertical arrows denote addition:

$$\begin{tikzcd}
W \times W \arrow[r, "f \times f"] \arrow[d] & V \times V \arrow[d] \\
W \arrow[r, "f"] & V
\end{tikzcd}$$

To do this we break the diagram into two parts. Recall that the addition map $V \times V \to V$ is defined by $V \times V \simeq F(k[M] \times_k F[M]) \to V$ where the latter map is the functorial image of (something akin to) addition of the two components. First we want to prove that the composition of the natural maps

$$W \times W \simeq (M \oplus M) \otimes T_1 \to F(k[M] \times_k F[M]) \to V \times V$$

is still $f \times f$. This can be checked by applying functoriality on the following two exact sequences corresponding to the two projections:

$$\begin{tikzcd} 0 \arrow[r] & M \oplus M \arrow[r] \arrow[d, "\pi_i"] & k[M] \times_k k[M] \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & k[M] \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

Therefore, we need only show that the following diagram commutes:

$$\begin{tikzcd} W \times W \arrow[r] \arrow[d] & F(k[M] \times_k k[M]) \arrow[d] \\ W \arrow[r] & V \end{tikzcd}$$

Which follows by functoriality applied to the following (note that it is different from the ones above):

$$\begin{tikzcd} 0 \arrow[r] & M \oplus M \arrow[r] \arrow[d, "add"] & k[M] \times_k k[M] \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M \arrow[r] & k[M] \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

This proves that $f : W \to V$ is linear. $\square$

In practice the main reason we want a tangent-obstruction theory for a pro-representable functor F is because we get dimension estimates on the pro-representing ring R. These dimension estimates are *extremely* useful. With that in mind, next week we will prove that if $F = h_R$ then

$$\dim T_1 - \dim T_2 \leq \dim R \leq \dim T_1$$

for any tangent-obstruction theory (T_1, T_2) of F.

Deformation Theory

Michael Kemeny*

Lecture 9

1 Deformations of a scheme

Conventions. By a ring we will mean a noetherian commutative ring with unity. By a scheme we will mean a locally noetherian, seperated scheme.

Definition 1.1. Let X be a scheme over an algebraically closed field k. Let $A \in \mathrm{Art}_k$. An *infinitesemal deformation of X over A* is a cartesian diagram

$$\begin{tikzcd} X \arrow[r, "\iota_Y"] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

where $Y \to \operatorname{Spec} A$ is flat.

Definition 1.2. An isomorphism of two infinitesimal deformations $Y, Y' \to \operatorname{Spec} A$ of X is an isomorphism $\varphi : Y \to Y'$ over A such that $\varphi \circ \iota_Y = \iota_{Y'}$. Hence, there is a commutative diagram as follows

$$\begin{tikzcd} & X \arrow[dl, "\iota_Y"'] \arrow[dr, "\iota_{Y'}"] & \\ Y \arrow[rr, "\varphi", "\sim"'] \arrow[dr] & & Y' \arrow[dl] \\ & \operatorname{Spec} A & \end{tikzcd}$$

Definition 1.3. Let $f : A \to B \in \mathrm{Art}_k$. The *pullback* of a deformation $Y \to \operatorname{Spec} A$ of X via f is defined to be $f^*Y \to \operatorname{Spec} B$ together with the map $f^*\iota_Y : X \to f^*Y$ induced by the universal property of the pullback.

Definition 1.4. We define the *functor of infinitesimal deformations of X* as $\mathrm{Def}_X \in \mathrm{Fun}^*(\mathrm{Art}_k, (\mathrm{Sets}))$ such that

$$\mathrm{Def}_X(A) = \{\text{Infinitesimal deformations of } X \text{ over } A\} / \sim$$

where the equivalence is that of isomorphism.

*TEX by Emre Sertöz. Please email suggestions or corrections to emresertoz@gmail.com

Definition 1.5. A deformation $(Y \to \operatorname{Spec} A, \iota_Y)$ of X is called trivial if it is isomorphic to the pullback of the "deformation" $(X \to \operatorname{Spec} k, \mathrm{Id}_X)$ via the structure morphism $s_A : \operatorname{Spec} A \to \operatorname{Spec} k$. A deformation $Y \to \operatorname{Spec} A$ of X is called *locally trivial* if there exists an open cover $\{U_i\}$ of Y such that U_i is a trivial deformation of $U_i \times_{\operatorname{Spec} A} \operatorname{Spec} k$.

There is another way to phrase local triviality. Observe that as a morphism of topological spaces $\iota_Y : |X| \to |Y|$ is a homeomorphism for any infinitesimal deformation (Y, ι_Y) (see the proof of the following lemma). Therefore, if U_i is an open subscheme in X then we can restrict the structure sheaf of Y to the open set $|U_i|$ to get an open subscheme U_i' of Y. Hence, an infinitesimal deformation (Y, ι_Y) is locally trivial iff there exists an open cover $\{U_i\}$ of X such that the induced open subscheme $U_i' \subset Y$ is a trivial deformation of U_i for each i.

Notation. We define the subfunctor Def_X^* of Def_X which gives the locally trivial deformations of X.

Technically speaking, we can skirt around the following lemma to prove the final theorem of this lecture as well as its corollary. However, we provide it here for completeness. Those of you who want to take the exam should keep in mind that **the following lemma is not part of the curriculum.**

Lemma 1.6 ([Ser06] pg. 23, Lemma 1.2.3)**.** *An infinitesimal deformation of a noetherian affine scheme is noetherian affine.*

Proof. Let $Z_0 = \operatorname{Spec} R_0$ be a noetherian affine scheme over k and $A \in \mathrm{Art}_k$. Consider an infinitesimal deformation of Z_0:

$$\begin{tikzcd} Z_0 \arrow[r, "j"] \arrow[d] & Z \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

Step 1

We will first show that j is a closed immersion with coherent nilpotent ideal of vanishing. This is local on the codomain so we will assume $Z = \operatorname{Spec} B$ with $A \to B$ flat. Then $Z_0 = \operatorname{Spec}(B \otimes_A k)$. Tensoring the exact sequence $0 \to \mathfrak{m}_A \to A \to k \to 0$ with the flat A-algebra B we get:

$$0 \longrightarrow \mathfrak{m}_A \otimes_A B \longrightarrow B \longrightarrow B \otimes_A k \longrightarrow 0$$

Since $\mathfrak{m}_A$ is nilpotent in A so is $\mathfrak{m}_A \otimes_A B$.

Step 2

We will show that j is a homeomorphism. This is local on the codomain so we may assume Z affine. A closed immersion of an affine scheme cut out by a nilpotent ideal is clearly a homeomorphism.

A quicker way to see this is to note $Z_0 \simeq V(N)$ as a subscheme of Z and that $|V(N)| = |Z|$ by definition of $V(N)$.

Step 3

If r is the smallest positive integer such that $N^r = 0$ then we have

$$Z = V(N^r) \supset V(N^{r-1}) \supset \cdots \supset V(N) = Z_0.$$

Therefore, our main result is true by induction on r if it is true when $r = 2$. Assuming $r = 2$ we note that N obtains the structure of an $\mathcal{O}_{Z_0}$-module. We use the homeomorphism j and the fact that cohomology of modules can instead be computed by treating them as $\mathbb{Z}$-modules to conclude that

$$\mathrm{H}^1(Z, N) = \mathrm{H}^1(Z_0, N) = 0$$

where the last equality follows because Z_0 is affine.

Consequently we have the following exact sequence, where $R := \mathrm{H}^0(Z, \mathcal{O}_Z)$:

$$0 \longrightarrow N(Z) \longrightarrow R \longrightarrow R_0 \longrightarrow 0$$

Let $Z' = \operatorname{Spec} R$, the exact sequence above gives us a map $Z_0 \to Z'$ which is a homeomorphism. Moreover the map $R \xrightarrow{=} \mathrm{H}^0(Z, \mathcal{O}_Z)$ gives us a map of schemes $\theta : Z \to Z'$ which fits into a commutative diagram as follows:

$$\begin{tikzcd} Z \arrow[rr, "\theta"] & & Z' \\ & Z \arrow[ul, "j"] \arrow[ur] & \end{tikzcd}$$

Since the two arrows from Z are homeomorphisms, so must θ. It remains to show that $\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is an isomorphism.

Step 4

Observe that Z is quasi-compact because θ identifies the underlying topological space with that of Z', which is affine hence quasi-compact. Moreover, because Z' is seperated, intersection of any two affine subschemes of Z' are affine and in particular quasi-compact. For any $f \in R$ note that the open sets $Z_f \subset Z$ and $D(f) \subset Z'$ coincide. In particular we have shown that for any $f, g \in R$ the intersection $Z_f \cap Z_g$ is quasi-compact. Now we may apply Ex. II.2.16d of [Har77] which states that $\Gamma(Z_f, \mathcal{O}_Z) \simeq R_f$. This implies that $\mathcal{O}_{Z'} \to \mathcal{O}_Z$ is an isomorphism. $\square$

Theorem 1.7. *Any infinitesimal deformation of a smooth affine scheme is trivial.*

Proof. Using Lemma 1.6 we reduce to deformations of algebras. Let B_0 be a smooth algebra over k and let $A \in \mathrm{Art}_k$. Consider a deformation of algebras

$$\begin{tikzcd} B_0 & B \arrow[l] \\ k \arrow[u] & A \arrow[l] \arrow[u] \end{tikzcd}$$

We wish to show that B is isomorphic to $B_0 \otimes_k A$ in a way that respects the map $B \otimes_A k \simeq B_0$.

We will do this by induction on $\dim A$. The base case $\dim A = 1$ implies $A = k$ and there is nothing to prove. If $\dim A \geq 2$, choose $0 \neq t \in \mathfrak{m}_A$ and consider the following principal small extension

$$0 \longrightarrow (t) \longrightarrow A \longrightarrow A' \longrightarrow 0$$

We have the following diagram

$$\begin{array}{ccccc} B & \longrightarrow & B \otimes_A A' & \longrightarrow & B_0 \\ \uparrow & & \uparrow & & \uparrow \\ A & \longrightarrow & A' & \longrightarrow & k \end{array}$$

By induction hypothesis $B \otimes_A A' \simeq B_0 \otimes_k A'$. There is a map $B_0 \otimes_k A \to B_0 \otimes_k A'$ and we wish to lift the map $B \to B_0 \otimes_k A'$ to $B_0 \otimes_k A$.

Since $A \to B$ is flat and the only geometric fiber of the map is smooth we conclude that B is smooth over A (see [Har77] Thm III.10.2). Moreover we have the following exact sequence

$$0 \longrightarrow B_0 \otimes_k (t) \longrightarrow B_0 \otimes_k A \longrightarrow B_0 \otimes_k A' \longrightarrow 0$$

where the kernel is a square zero ideal because $(t) \subset A$ is. Therefore we may use the smoothness criterion ([Ser06] Theorem C.9) to conclude that

$$\hom_A(B, B_0 \otimes_k A) \to \hom_A(B, B_0 \otimes_k A')$$

is surjective. In particular, the lift we are after exists.

Exercise. Compare the definition of smoothness for functors with this lifting property of smooth algebras.

It remains to show that the map $B \to B_0 \otimes_k A$ is an isomorphism. Clearly $B_0 \otimes_k A$ is free and thus flat over A, the map is an isomorphism on the special fiber from which it follows that the map itself is an isomorphism by Lemma 3.1 of Lecture 5 (or [Ser06] Lemma A.4). $\square$

We will now study locally trivial deformations. Something that is locally trivial is formed by gluing trivial objects by isomorphisms. Therefore the following lemma will be crucial for our understanding of the functor Def^*_X. Note however that this is a simplified version of [Ser06] Lemma 1.2.6. We will eventually need the stronger version but this will do for now.

Lemma 1.8. *Let A be a k-algebra and $A[\varepsilon] = A \times_k k[\varepsilon]$. Denote by $\mathrm{Aut}^*_{k[\varepsilon]}(A[\varepsilon])$ the group of $k[\varepsilon]$-algebra automorphisms of $A[\varepsilon]$ that induce the identity on $A[\varepsilon]/(\varepsilon) \simeq A$. Denote by $\mathrm{Der}_k(A)$ the k-derivations of A into itself.*

There is a natural isomorphism of groups

$$\mathrm{Aut}^*_{k[\varepsilon]}(A[\varepsilon]) \xrightarrow{\sim} \mathrm{Der}_k(A).$$

Proof. Since there is a natural splitting $A[\varepsilon] = A \oplus \varepsilon A$ as a k-vector space, we may define two projections from $A[\varepsilon]$ to A which we will denote by π_1 and π_2, such that the first projection is a morhpism of algebras whereas the second projection is only k-linear. Letting $\psi \in \mathrm{Aut}^*_{k[\varepsilon]}(A[\varepsilon])$, we may write $\psi = \psi_1 + \varepsilon \cdot \psi_2$ where $\psi_i = \pi_i \circ \psi$.

By $k[\varepsilon]$-linearity we have $\psi(a + \varepsilon b) = \psi(a) + \varepsilon\psi(b)$. Expanding this out gives the following identities:

$$\begin{aligned} \psi_1(a + \varepsilon b) &= \psi_1(a) \\ \psi_2(a + \varepsilon b) &= \psi_2(a) + \psi_1(b). \end{aligned}$$

Finally we combine this with our assumption that ψ descends to the identity on A to conclude $\psi_1 = \pi_1$. Therefore, given our restrictions on ψ only $\mathrm{d} := \psi_2|_A$ is not completely determined.

We now prove that d is a k-derivation of A. Indeed, for $c \in k$ we have $\psi(c) = c\psi(1)$ and thus $\mathrm{d}(c) = 0$. If $a, b \in A$ then expanding out $\psi(ab) = \psi(a)\psi(b)$ gives $\mathrm{d}(ab) = a\,\mathrm{d}(b) + b\,\mathrm{d}(a)$.

Conversely, begin with a k-derivation $\mathrm{d} : A \to A$ and define

$$\psi := \pi_1 + \varepsilon(\mathrm{d} + \pi_2) : A[\varepsilon] \to A[\varepsilon].$$

It is left as an exercise to show that ψ is a $k[\varepsilon]$-algebra automorphism.

We have thus established a bijection between automorphisms of $A[\varepsilon]$ and k-derivations of A. It is clear that this bijection respects the group structure and is thus a group isomorphism. $\square$

Let X be a scheme, then we denote $\mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)$ by $\mathcal{T}_X$ and call it *the tangent bundle of X*.

Theorem 1.9. *Let X be a variety over k (ie. integral, seperated scheme of finite type over k). Then the tangent space of the functor of locally trivial deformations of X is canonically isomorphic to $\mathrm{H}^1(X, \mathcal{T}_X)$.*

Proof.

Step 1: From locally trivial families to cohomology classes

Let $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$ be a locally trivial infinitesimal deformation of X. Choose an affine open cover $\{U_i \to X\}$ such that $\mathcal{X}|_{U_i}$ is a trivial deformation of U_i. Choose isomorphisms $\theta_i : U_i \times \operatorname{Spec} k[\varepsilon] \to \mathcal{X}|_{U_i}$ and by pulling back these isomorphisms we define automorphisms $\theta_{ij} := \theta_i^{-1}\theta_j : U_{ij} \times \operatorname{Spec} k[\varepsilon] \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$ where $U_{ij} := U_i \cap U_j = U_i \times_X U_j$.

Each U_{ij} is affine because X is seperated and thus Lemma 1.8 corresponds to each automorphisms θ_{ij} a derivations of $\Gamma(U_{ij}, \mathcal{O}_X)$, equivalently an element $d_{ij} \in \Gamma(U_{ij}, \mathcal{T}_X)$. Since $\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = \mathrm{Id}$ we conclude $d_{ij} + d_{jk} - d_{ik} = 0$. Thus $\{d_{ij}\}$ is a Check 1-cocycle and therefore gives a class in $\mathrm{H}^1(X, \mathcal{T}_X)$.

Step 2: Showing invariance on the isomorphism class

We made three choices in constructing the cohomology class from the isomorphism class of a locally trivial family. First we chose a representing family, second we chose a trivializing cover $\{U_i\}$ and then we chose trivializing isomorphisms θ_i. With one stroke we can show the invariance of the resulting cohomology class on the first and third choices whilst the second one is an easy exercise:

Exercise. Show the cohomology class does not depend on the trivializing cover $\{U_i\}$.

Let $\mathcal{X}' \to \operatorname{Spec} k[\varepsilon]$ be another deformation and let $\Psi : \mathcal{X}' \to \mathcal{X}$ be an isomorphism of deformations. The cover $\{U_i \to X\}$ also trivializes $\mathcal{X}'$ and therefore we choose trivializing isomorphisms $\theta'_i : U_i \times \operatorname{Spec} k[\varepsilon] \to \mathcal{X}'|_{U_i}$. Let $d'_{ij} \in \Gamma(U_{ij}, \mathcal{T}_X)$ be constructed as above, this time using θ'_i's. Let $\mu_i : U_i \times \operatorname{Spec} k[\varepsilon] \to U_i \times \operatorname{Spec} k[\varepsilon]$ be the isomorphism given by $\theta_i^{-1}\Psi\theta'_i$. Observe the following:

$$\begin{aligned}
\theta_i'^{-1}\theta'_j &= \theta_i'^{-1}\Psi\Psi^{-1}\theta'_j \\
&= (\theta_i\mu_i)^{-1}(\theta_j\mu_j) \\
&= \mu_i^{-1}\theta_i^{-1}\theta_j\mu_j
\end{aligned}$$

Therefore, letting $u_i \in \Gamma(U_i, \mathcal{T}_X)$ be the derivation associated to μ_i we get $d'_{ij} = d_{ij} + u_j - u_i$. In other words, the cohomology classes associated to $\{d'_{ij}\}$ and $\{d_{ij}\}$ are equal.

If we take $\mathcal{X}' = \mathcal{X}$ and $\Psi = \mathrm{Id}$ then this also proves the independence of the class on the choice of trivializing isomorphisms θ_i.

Step 3: From cohomology classes to deformations

Now we would like to reverse the operation and get a locally trivial deformation of X from a cohomology class $\xi \in \mathrm{H}^1(X, \mathcal{T}_X)$. Choose an affine open cover $\{U_i \to X\}$. By [Har77] Theorem III.4.5 we may calculate the sheaf cohomology using Čheck cohomology with the cover $\{U_i\}$. Thus $\xi = \{d_{ij}\}$ with $d_{ij} \in \Gamma(U_{ij}, \mathcal{T}_X)$ and $U_{ij} = U_i \cap U_j$ as before.

By Lemma 1.8 each d_{ij} corresponds to an isomorphism $\theta_{ij} : U_{ij} \times \operatorname{Spec} k[\varepsilon] \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$. We glue the schemes $U_i \times \operatorname{Spec} k[\varepsilon]$ along the open sets $U_{ij} \times \operatorname{Spec} k[\varepsilon]$ via these isomorphisms θ_{ij} to obtain a scheme $\mathcal{X}$ (see [Har77] Ex. II.2.12).

Since the automorphisms θ_{ij} don't interfere with the projection onto $\operatorname{Spec} k[\varepsilon]$ we can glue these projections to get a map $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$. Similarly, the maps $U_i \to \mathcal{X}$ glue to a map $X \to \mathcal{X}$ because each automorphism θ_{ij} is an automorphism of deformations, therefore it fixes the special fiber $U_{ij} \to U_{ij} \times \operatorname{Spec} k[\varepsilon]$. This gives us the following diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec} k \arrow[r] & \operatorname{Spec} k[\varepsilon]
\end{tikzcd}$$

The map $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$ is flat because flatness is local on the domain and locally this map is just the projection from a trivial family. Moreover, the map $\mathcal{X} \to \operatorname{Spec} k[\varepsilon]$ is proper because $X \to \operatorname{Spec} k$ is proper (since any morphism from $k[\varepsilon]$ to a DVR kills ε the implication is immediate).

Step 4: Invariance on the representative of the cohomology class

We would like to show now that the isomorphism class of the deformation constructed from $\xi \in \mathrm{H}^1(X, \mathcal{T}_X)$ is invariant with respect to the choice of a representative in ξ. This requires first that we show invariance under refinements of the cover we use for Čheck cohomology, which is easy, and then invariance for two different representatives $\{d_{ij}\}, \{d'_{ij}\} \in \xi$. Let $\mathcal{X}$ correspond to $\{d_{ij}\}$ and $\mathcal{X}'$ correspond to $\{d'_{ij}\}$. We will show that these two deformations are isomorphic. To do this, use the fact that $\{d_{ij} - d'_{ij}\} = \{a_j - a_i\}$ where $a_i \in \Gamma(U_i, \mathcal{T}_X)$. The a_i's correspond to isomorphisms $\alpha_i : U_i \times \operatorname{Spec} k[\varepsilon] \to U_i \times \operatorname{Spec} k[\varepsilon]$ and these α_i's glue to an isomorphism $\mathcal{X} \to \mathcal{X}'$. □

Using Theorem 1.7 together with Theorem 1.9 we have the following.

Corollary 1.10. *If X is a smooth variety over k then its first order deformations are in bijection with* $\mathrm{H}^1(X, \mathcal{T}_X)$.

References

[Har77] R. Hartshorne, *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496, Graduate Texts in Mathematics, No. 52.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, vol. 334, pp. xii+339.

Deformation Theory

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Lecture 10

If X is a smooth variety over k then we know that $\mathrm{Def}_X(k[\varepsilon]) \simeq H^1(X, \mathcal{T}_X)$. We will give a similar expression for the tangent space of the deformation functor for any (integral) variety.

1 Background on extensions

The main reference for this section is Section III.6 of [Har77].

Let X be a scheme over k and $\mathrm{Mod}(X)$ be the category of $\mathcal{O}_X$-modules. An object $\mathcal{I}$ of $\mathrm{Mod}(X)$ is called *injective* if the functor $\mathrm{Hom}_X(-, \mathcal{I})$ is exact. Recall that $\mathrm{Mod}(X)$ has enough injectives. This means that for any object $\mathcal{F} \in \mathrm{Mod}(X)$ we can find an exact sequence (called an injective resolution of $\mathcal{F}$):

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^0 \longrightarrow \mathcal{I}^1 \longrightarrow \mathcal{I}^2 \longrightarrow \dots$$

where $\mathcal{I}^n \in \mathrm{Mod}(X)$ is injective for all $n \geq 0$.

Fix $\mathcal{G} \in \mathrm{Mod}(X)$. We will define $\mathrm{Ext}^i_X(\mathcal{G}, -)$ to be the i^{th} right derived functor of $\mathrm{Hom}_X(\mathcal{G}, -)$.

Using the resolution above $\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{F})$ is, by definition, the i^{th} cohomology of the following complex:

$$0 \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^0) \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^1) \longrightarrow \mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^2) \longrightarrow \dots$$

where $\mathrm{Hom}_X(\mathcal{G}, \mathcal{I}^n)$ is considered to be the n^{th} term of the complex. Observe that $\mathrm{Ext}^0_X(\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_X(\mathcal{G}, \mathcal{F})$.

If X is a scheme over k then $\mathrm{Ext}^i_X(\mathcal{G}, \mathcal{F})$ has a natural k-vector space structure. Indeed, in general the set of homomorphisms between two $\mathcal{O}_X$-modules has the natural structure of a $\Gamma(X, \mathcal{O}_X)$-module. Thus, so will their Ext.

1.1 Useful exercise

Exercise III.6.1 in [Har77] gives a concrete description of $\mathrm{Ext}^1_X(\mathcal{G}, \mathcal{F})$. An exact sequence

$$\xi : \qquad 0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{H} \longrightarrow \mathcal{G} \longrightarrow 0$$

is called *an extension of $\mathcal{G}$ by $\mathcal{F}$*. Two extensions are considered isomorphic if there is an isomorphism of short exact sequences that is an equality on $\mathcal{F}$ and $\mathcal{G}$. There is a natural k-vector space structure on the set of isomorphism classes of extensions. To get an idea on how this structure is defined see p.12-13 of [Ser06].

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Using the extension ξ we get a long exact sequence:

$$0 \to \mathrm{Hom}_X(\mathcal{G},\mathcal{F}) \to \mathrm{Hom}_X(\mathcal{G},\mathcal{H}) \to \mathrm{Hom}_X(\mathcal{G},\mathcal{G}) \xrightarrow{\delta} \mathrm{Ext}^1_X(\mathcal{G},\mathcal{F}) \to \dots$$

Define $[\xi] := \delta(\mathrm{Id}_{\mathcal{G}}) \in \mathrm{Ext}^1_X(\mathcal{G},\mathcal{F})$. This gives a bijection (and even a k-vector space isomorphism) between isomorphism classes of extensions of $\mathcal{G}$ by $\mathcal{F}$ and elements of $\mathrm{Ext}^1_X(\mathcal{G},\mathcal{F})$.

1.2 Ext sheaves

Let $\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{F})$ denote the hom-sheaf which is a $\mathcal{O}_X$-module. Do not confuse this with the $\Gamma(X,\mathcal{O}_X)$-module $\mathrm{Hom}_X(\mathcal{G},\mathcal{F})$.

The functor $\mathscr{H}\mathrm{om}(\mathcal{G},-)$ gives a left-exact functor from $\mathrm{Mod}(X)$ to itself and its i^{th}-right derived functor is denoted by $\mathscr{E}\mathrm{xt}^i_X(\mathcal{G},-)$.

If X is a variety then there is a relation between the cohomology groups of the sheaves $\mathscr{E}\mathrm{xt}^i_X$ with the modules Ext^i_X. This relationship is given by a spectral sequence called *local-to-global* Ext-*sequence*. In particular, it gives the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}^1(X,\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{F})) \arrow[r] & \mathrm{Ext}^1_X(\mathcal{G},\mathcal{F}) \arrow[r] & \mathrm{H}^0(X,\mathscr{E}\mathrm{xt}^1_X(\mathcal{G},\mathcal{F})) \arrow[dll, rounded corners] \\ & \mathrm{H}^2(X,\mathscr{H}\mathrm{om}(\mathcal{G},\mathcal{F})) \arrow[r] & \mathrm{Ext}^2_X(\mathcal{G},\mathcal{F}) & \end{tikzcd}$$

See Huybrecths "Fourier-Mukari transforms in algebraic geometry" section 2.3 or the Wikipedia page on the "Grothendieck spectral sequence" and the references therein (also see the page on "Five term exact sequence").

2 Main theorem

We are now in a position to state the main theorem we are aiming for.

Theorem 2.1. *Let X be a variety over k (in particular X is integral). Then*

$$\mathrm{Def}_X(k[\varepsilon]) \simeq \mathrm{Ext}_X(\Omega^1_X,\mathcal{O}_X).$$

Remember that $\mathcal{T}_X = \mathscr{H}\mathrm{om}(\Omega^1_X,\mathcal{O}_X)$, therefore the first part of the local-to-global Ext-sequence gives us

$$0 \longrightarrow \mathrm{H}^1(X,\mathcal{T}_X) \longrightarrow \mathrm{Ext}^1_X(\Omega^1_X,\mathcal{O}_X)$$

We proved that the first term parametrizes the first order *locally trivial* deformations of X. This injection corresponds to the inclusion of locally trivial deformation functor into the deformation functor.

2.1 Preliminaries for the main theorem

We need two lemmas before we start proving the theorem next week. The first gives a criterion for flatness over artinian local rings.

Remark. It is well known that a *finite* module over a local ring is flat iff it is free. The following is a stronger result, eliminating the finiteness condition, for Artinian rings. Furthermore, we give a simple criterion to check for flatness.

Lemma 2.2. *Let $A \in \mathrm{Art}_k$ and let M be an A-module. Then M is flat iff*

$$\mathrm{Tor}_1^A(M, k) = 0.$$

Moreover, if M is flat then it is free.

Proof.
($\Longrightarrow$) If M is flat then for any ideal $I \subset A$ we have $\mathrm{Tor}_1^A(M, A/I) = 0$. In particular for $I = \mathfrak{m}_A$.

($\Longleftarrow$) We are going to show that M is free and hence flat.

Consider the following exact sequence:

$$0 \longrightarrow \mathfrak{m}_A \longrightarrow A \longrightarrow k \longrightarrow 0 \tag{1}$$

Let $\{x_i\} \subset M$ be such that the images of this subset forms a basis in $M \otimes_A k$. Using Nakayama's lemma for nilpotent maximal ideals we conclude that $\{x_i\}$ generate M.

Suppose there is a relation amongst $\{x_i\}$, i.e. there exists $\{f_i\} \subset A$ such that $\sum f_i x_i = 0$ (note that almost all terms have to be zero for the sum to make sense). We wish to show that in fact all f_i are zero. Since x_i restrict to a basis on $M \otimes k$, the f_i's must vanish under the map $A \to k$. Therefore, for any relation $\{f_i\}$, each f_i belongs to $\mathfrak{m}_A$.

We are now going to show that if all relations belong to $\mathfrak{m}_A^j$ then in fact all relations belong to $\mathfrak{m}_A^{j+1}$. Since $\mathfrak{m}_A$ is a nilpotent ideal, this means that there are no non-zero relations amongst $\{x_i\}$ and that M is free with generators $\{x_i\}$.

We established the base case $j = 1$ above. (If we take the obvious $j = 0$ to be the base case then the induction step below will not work. Why?) Observe that if all relations belong to $\mathfrak{m}_A^j$ then $\{x_i\}$ restricts to a free basis of $M/\mathfrak{m}_A^j M$.

Take a relation $\{f_i\} \subset \mathfrak{m}_A^j$ and consider the natural map

$$\mathfrak{m}_A \otimes_A M \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^{j+1} \otimes_{A/\mathfrak{m}_A^j} M/\mathfrak{m}_A^j$$

If $\sum_i f_i \otimes x_i = 0$ then its image under this map must also be zero. But x_i maps to a free basis on $M/\mathfrak{m}_A^j$ so this implies that the image of each f_i in $\mathfrak{m}_A/\mathfrak{m}_A^{j+1}$ must be zero, hence $f_i \in \mathfrak{m}_A^{j+1}$. So we need only show that for any relation $\sum_i f_i x_i = 0$ the tensor $\sum_i f_i \otimes x_i$ is also zero.

To see this, tensor the equation (1) with M over A. Using the fact that $\mathrm{Tor}_1^A(M, k) = 0$ we conclude $\mathfrak{m}_A \otimes_A M \to \mathfrak{m}_A M$ is an isomorphism. □

The following lemma may be considered as a flatness criterion for "thickenings" of deformations via principal small extensions.

Lemma 2.3. *Suppose $A \to B \in \mathrm{Art}_k$ is a principal small extension. Let $f : A \to C$ be a morphism of k-algebras. The morphism f is flat iff the following two conditions are satisfied:*

- *$f' : B \to B \otimes_A C$ is flat*
- *$\ker(C \to B \otimes_A C) \simeq k \otimes_A C$.*

Proof.

($\Longrightarrow$) Flatness is preserved under base change. Hence, if f is flat so is f'. Tensor the exact sequence $0 \to k \to A \to B \to 0$ (treated as a sequence of A-modules) with the flat A-module C to get

$$0 \longrightarrow k \otimes_A C \longrightarrow C \longrightarrow B \otimes_A C \longrightarrow 0$$

This gives us the second condition.

($\Longleftarrow$) Now assume the two listed conditions are satisfied. The second of the conditions imply that $\mathrm{Tor}_1^A(B, C) = 0$ as the first map in the following exact sequence is injective:

$$k \otimes_A C \longrightarrow C \longrightarrow B \otimes_A C \longrightarrow 0$$

Now tensor the exact sequence $0 \to \mathfrak{m}_B \to B \to k \to 0$ seen as A-modules with C:

$$\mathfrak{m}_B \otimes_A C \longrightarrow B \otimes_A C \longrightarrow k \otimes_A C \longrightarrow 0$$

We could have viewed the exact sequence as B-modules and tensored with $B \otimes_A C$ instead, to get the same exact sequence. Since $B \otimes_A C$ is flat over B, the map $\mathfrak{m}_B \otimes_A C \to B \otimes_A C$ is injective. Using the Tor-exact sequence we conclude that $\mathrm{Tor}_1^A(C, B) \to \mathrm{Tor}_1^A(C, k)$ is surjective. But $\mathrm{Tor}_1^A(C, B) \simeq \mathrm{Tor}_1^A(B, C) = 0$ and hence $\mathrm{Tor}_1^A(C, k) = 0$. Applying the previous lemma finishes the proof. $\square$

References

[Har77] R. Hartshorne, *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977, pp. xvi+496, Graduate Texts in Mathematics, No. 52.

[Ser06] E. Sernesi, *Deformations of algebraic schemes*, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 2006, vol. 334, pp. xii+339.